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БИБЛИОТЕКА СТУДЕНТА

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INTRODUCTION

No matter what historical civilisation he belongs to, man has a need for truth. The pursuit of truthful knowledge is common both to human beings from primitive civilisations and to our contemporaries as they perceive the world around them. This knowledge brings joy and satisfaction to some, and quite the opposite, sadness, to others. For the strong, truth represents a call to heroic deeds, whilst the weak are paralysed in their will power and plunged into pessimism and perplexity. But for all this, everyone without exception seeks to understand the world in which they live.

It is, however, a far from simple task to attain truthful knowledge, even that which is incomplete and imperfect. Sometimes it involves self-sacrifice. The Italian scholar and philosopher Giordano Bruno, following Copernicus' heliocentric cosmology, put forward a concept of the infinity and innumerable multitude of worlds in the universe, only to be accused of heresy and burned to death by the Roman inquisition. A number of the physicists who studied radioactivity were exposed to irradiation. Some microbiologists even went as far as to carry out life-endangering experiments on themselves.

People wish to know not only the laws of nature and the essence of social phenomena, but also the secrets of the human brain. As early as the 17th century, the British philosopher Francis Bacon was moved to declare that knowledge and power were the same thing.

However, the road to truth is a thorny one indeed. The great philosopher Karl Marx wrote, "There is no royal road to science, and only those who do not dread the fatiguing climb of its steep paths have a chance of gaining its luminous summits."[1]

In order to extend the reach of his knowledge, man created the microscope and the telescope, radio and television, the computer and the space rocket, which enabled him to gain a more profound understanding of natural and social phenomena.

Various means of gaining knowledge have also been developed, such as modelling and the application of mathematical methods, physical and biological experiments, genetic engineering and electronic data processing.

If effective use is to be made of all these methods and inventions, human thinking must be logically correct. Laws of development apply to nature, to society, and, of course, to thought itself. It is one of man's specific features that he aspires to perceive the laws of correct thinking, i.e., logical laws. He is assisted in this perception by the science of logic.

Can man think correctly even if he does not know the rules and laws of logic, but just employs them on an intuitive level? After all, there are musicians who are quite able to play an instrument without being versed in musical notation. But such musicians are limited in their range. They are unable to perform works written down in the form of notes, or to record the melody they compose. A person with a mastery of logic thinks in clearer terms, his arguments are more precise and carry greater weight, and he is less prone to error.

Logical thinking is not inborn, but can and must be developed in various ways. The systematic study of the science of logic is one of the most effective means to develop logical abstract thinking.

An interesting method of the development of thought is the resolution of logical problems. The American mathematician Raymond M. Smullyan has devised a large number of instructive problems in this context. Let us examine one of them. A man was accused of a robbery. Present in court were a prosecuting

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[1] K. Marx, *Capital*, Vol. 1, Progress Publishers, Moscow, 1978, p. 30.

counsel and one for the defence. The prosecuting lawyer stated, "If the accused is guilty, then he had an accomplice." To which the defence counsel replied, "It's not true." He could have said nothing worse. In so doing, he not only admitted the guilt of the defendant, but laid the sole responsibility for the crime on his shoulders, thus exposing him to a greater degree of punishment. The defence counsel made this mistake because he was unable to correctly formulate his thoughts.

Thought and language determine each other. It is no coincidence that the emergence of logic as a science was associated with rhetoric, the teaching of oration. Logic first appeared in Ancient Greece and Ancient India. Public contests between orators were very popular there and attended by large audiences. V. Vasiliev, the famous Russian orientalist, comments as follows on the contests in Ancient India. If anyone appeared and started to expound ideas that were previously quite unknown, he would not be shunned and punished without any trial. On the contrary, he would be readily welcomed provided he was able to satisfy all objections and disprove the old theories. An arena was erected for the contest, judges chosen and the emperors, nobility and commoners were present at the contest. If the contest was a duel, the defeated party would sometimes have to take his own life by plunging into the river or leaping from the cliffs, or else become a slave of the victor and enter into his faith. If the vanquished party enjoyed respect, being, for instance, the monarch's teacher, and thus amassed a large fortune, his possessions were often given to a man in rags, his victorious opponent. However, the contest was frequently not limited to individuals and involved entire factions.

In our times, disputes (debates, discussions) assume a different form, but in terms of their essence and content are much fiercer. Let us recall the 1987 international forum in Moscow entitled, "For a Nuclear-free World, for the Survival of Mankind". The issue being discussed was no private concern, but one which affects both each individual living on earth and the human race as a whole, namely the survival of mankind and the maintenance of civilisation.

Truth and logic are interlinked. Logic helps to prove truthful judgements and to disprove false ones, it teaches people to think clearly, concisely and correctly. Logic is needed by all, whatever their occupation may be. It is required by teachers, since they are unable to effectively develop their pupils' thinking if they lack a mastery of logic. Lawyers build their cases for the prosecution or the defence by using logic. Doctors diagnose illnesses on the basis of the symptoms. Logic is thus required by everyone, whether he may work by brain or by hand.

Logic assists students to assimilate the diversity of information with which they are confronted both in the study of various disciplines and in their practical work. Subsequently, in the course of continued self-education, logic helps them to distinguish the important from the trivial, to take a critical view of the definitions given in various books and the classification of different concepts, to find ways to prove their truthful judgements and to disprove false ones. These are just some of the many advantages to be gained from a study of the extremely interesting and age-old science of logic, the science of the laws and forms of correct thought.

Apart from the main forms of correct thought – concepts, judgements, inferences – this book also covers methods of demonstration and refutation, some of the many types of logical errors encountered in thinking, various forms of hypotheses, etc. The symbols of mathematical logic are used in some sections. In the last chapter, readers will be acquainted with the history of the development of classical logic and the main trends in present-day symbolic logic.

This book is designed to help readers develop a disciplined way of thinking. And an ability to think creatively is a skill required by everyone.

Chapter I

THE SUBJECT AND MEANING OF LOGIC

The term "logic" is derived from the Greek word *logos*, which means "thought", "word", "reason," "law" and is used to denote the totality of rules to which the process of thought is subjected, a process that reflects reality. It is also used to denote the science of the rules of reasoning and the forms in which it occurs. We shall use the term "logic" in these two senses. The term is also employed to denote the laws governing the objective world ("logic of things", "logic of events"). This sense of the term goes beyond the bounds of the book before you.

Thought is a subject of study not only on the part of logic, but also by a number of other disciplines such as psychology, cybernetics, educational science. Each of them studies thought in a way that is specific to it alone. Psychology studies thought from the angle of the motives which evoke it, revealing the individual peculiarities of thought. Cybernetics is interested in aspects of thought which are associated with the rapid and efficient data processing, the link between thought and language (natural and artificial), methods and systems of programming, preparation of computer software, and a number of other issues. The educational sciences study thought as a process of cognition in the course of learning and teaching. It is the physiological foundations of thought which are of interest to the physiology of higher nervous activity, such as the processes of excitation and inhibition taking place in the human brain.

Logic examines thought as a means of cognising the objective world, those of its forms and laws in which the world is reflected in the process of thought. Since the process by which the world is cognised in its entirety is the subject of philosophical studies, logic represents a philosophical science.

Cognition does not exist in a single state, like something static, but is progress towards an objective, complete and all-embracing truth. This process comprises many elements and aspects, which are necessarily linked to one another.

Materialist dialectics, in revealing the content of the various aspects of cognition, establishes their interaction and the role they play in the attainment of truth. The social nature of cognition and the active character of human cognitive activity are analysed from the viewpoint of dialectical and historical materialism. Thought is examined in the context of an understanding of truth (objective, absolute, and relative) as also with regard to the study of methods and forms of scientific cognition (such as axiomatic methods, methods of formalisation, mathematical methods, probabilistic methods, modelling, and a number of others).

In order to reveal the importance of logic as a science more fully, it will be necessary to examine thought as the subject which it studies.

§ 1. Thought as the Subject Studied by Logic

Cognition as a reflection of reality

Cognition is a dialectical process by which the material world is reflected in human consciousness. It is the movement of thought from ignorance to knowledge, from incomplete and inaccurate knowledge to fuller and more precise forms.

People do not cognise the world by virtue of an inborn inquisitiveness. The cognition of the world derives from the need to change it. Materialists, representatives of a philosophical trend which holds that matter is primary and consciousness – a quality displayed by highly organised matter (the human brain) – is secondary, consider that the world and the laws governing its development are cognisable.

The scientific theory of cognition is the theory of reflection. Its essential meaning is the following. Material things exist outside our consciousness. Due to the effect exerted by these things, images, or "moulds", "photographs", "copies" of objects are formed in the human mind. Images cannot exist without actual objects (in the absence of aircraft, for example, there would be no images of such objects), but things exist objectively, regardless of their images (for example, jungle plants and birds exist even if nobody has seen them, i.e., even if their images are absent from human consciousness). Images of things accord with the things whose depiction they represent (the image of an elephant in my consciousness, for example, and an actual elephant resemble each other). Images thus have a cognitive significance. The image is ideal, for it cannot exist outside human consciousness. But the image and the thing it represents are not entirely identical. This is why we can watch a film on two separate occasions and notice something on the second which we have failed to perceive on the first. We may look at a work of art many times and still keep finding something we have failed to notice previously. The image is thus inferior to the actual thing, and we are unable to make it encompass all details attributable to the thing. The thing and its qualities reveal themselves to us in the process of cognition.

The foundation of cognition is practice. Practice is the motive, the generative force of cognition and the criterion of truth. In their activities, people come up against various qualities of things and phenomena which they are unable to understand. If they are to obtain material benefits, they must study nature and know its secrets. Cognition of the qualities of objects is essential for putting nature at the service of man. The study of the structure of the atomic nucleus, for example, enabled man to find a new source of energy. In the jungles of South America there lives a tiny frog. Having a length of just one to three centimetres and a weight of one gram, it nevertheless is able to store enough poison to kill 50 jaguars. One of the Indian tribes which has no firearms continues to use poison arrows. They obtain the poison they need from this frog. It is the most powerful poison of animal origin known to date. This example shows that a practical requirement on the part of the Indians leads them to take cognisance of the property exhibited by this frog's poison. Further, the practical need to cure a number of illnesses has led people to discover that in small doses this same poison can be used as a curative.

At the present time, practice has confronted mankind with a number of global problems: the conservation of nature on our planet, the development of new sources of energy, the exploration and use of outer space, the resources of the world's oceans, etc. Cognition is geared to resolving these issues as well.

It may be said that all sciences ultimately derive from man's practical requirements: mathematics from the need to measure plots of land and the volume of vessels, astronomy from maritime needs, medicine from the need to fight disease, and Marxism arose from a need to find answers to the questions thrown up by the proletariat's class struggle against the bourgeoisie. The requirements arising from the development of agriculture and industry confronted chemistry with a number of tasks: the provision of dependable and cheap raw materials, fabrics, building materials, mineral resources, etc. And chemists met these challenges. Cognition and all sciences develop then on the basis of practical requirements.

How does the process of cognition take place? "From living perception to abstract thought, *and from this to practice*,– such is the dialectical path of the cognition of *truth*, of the cognition of objective reality."[1] Cognition occurs in two main forms – *sensory perception* and *abstract thought*. Practice is not a separate form, since the process of cognition begins with practice (as the basis of cognition) and ends with practice (as the criterion of truth).

All acts of cognition proceed from observation, from sensations, sensory perceptions. Objects exert an effect on our sense organs and give rise to sensations and perceptions in the brain. Man has no other means but his sense organs for receiving signals from the outside world and transmitting them to the brain.

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[1] V. I. Lenin, "Conspectus of Hegel's Book *The Science of Logic*", *Collected Works*, Vol. 38, Progress Publishers, Moscow, 1980, p. 171.

Sensation, perception and representation are three forms of sensory cognition. *Sensation* is the reflection of individual properties of objects or phenomena belonging to the material world and acting directly on the sense organs (for example, the reflection of bitter, salty, hot, red, round, smooth, etc., properties).

Each object has not one, but many properties. Sensations do indeed reflect the various properties of objects. Sensations as the subjective image of an objective world arise in the cortex. The sensitivity of the sense organs may be heightened by training. Laymen will usually distinguish three or four shades of black, whereas experts may see up to 40 shades.

Sensation represents a direct link between consciousness and the outside world. Sensations arise due to the effect of objects on the various sense organs – sight, hearing, smell, touch, taste. If anyone is deprived of one or more of his sense organs (if he is deaf and blind, for example), the remaining sense organs are greatly sharpened and partially fulfil the functions of those which are lacking. *The Miracle Worker* by William Gibson is a play about the childhood and schooldays of Helen Keller, an American deaf and blind girl. It vividly expresses the immense difficulties of communication with Helen and the method by which she was taught. When the little girl spoke her first word, it was considered a miracle. She learned to speak even though she could not hear her own voice.

Perception is the integral reflection of an external material object acting directly on the sense organs (for example, the image of a bus, a wheat field, a power station, a book, etc.). Perception is a process composed of sensations. The perception of an orange is, for example, made up of sensations referring to its spherical shape, its orange colour, its sweetness, aroma and others. Though perceptions represent a sensory image reflecting an object which has an impact on an individual at a particular time, they depend to a great extent on past experience. The completeness with which a green meadow is perceived, for example, will differ between a child, an adult, an artist, a biologist and a farmer (the first will admire its beauty, the biologist will see in it various species of plants and the farmer will be interested in how much grass and hay it will yield, etc.).

The perception of objects very frequently overlaps closely with thought. The extent to which perceptions depend on previous experience and knowledge can be seen from the following story. It is said that a European touring Central Africa made a stop in a small village whose inhabitants had no idea about books and newspapers. As his horses were being changed, he opened a newspaper and began to read it. A crowd gathered around him and watched his actions closely. When the traveller was preparing to continue his journey, the local people came up to him and asked him to sell the newspaper for a large sum of money. To the traveller's question as to why they needed the newspaper, they said that they had seen him looking for a long time at its black patterns. This had obviously healed his eyes and they, too, would like to have such a remedy. The villagers, having no idea what reading was, based their judgement on their previous experience and thus perceived the newspaper to be a means of healing.

Representation is the sensuous image of an object which we cannot perceive at the moment in question, but did perceive in one form or another at some time in the past. A representation can be reproductive (for example, everyone has an image of their home, place of work, images of friends and relatives which they do not see at a particular moment in time). It can also be creative, and sometimes extend into the realm of fantasy. Creative imagination in man may arise as a result of literary description. He may, for example, imagine the tundra or the jungle from a description, although he has never been there, and he may also imagine the Northern Lights even though he has never been to the Far North and seen them with his own eyes.

We use a description of the outward appearance of an actual person or a character from literature to conjure up his image in our mind's eye and imagine what he looks like. Let us give an example. We may recall a scene from *The Count*, a film with Charlie Chaplin. Charlie, the imaginary count, found himself in a difficult situation. When a large piece of water-melon was placed before him, he attacked it through ignorance without a knife or fork. As was only to be expected, it soon became rather uncomfortable to gnaw at the flesh of the fruit. The hard and sharp edges of the peel even found their way into his ears. In order to avoid this, Charlie wrapped a serviette around his cheeks. This is comic in itself, for what could be simpler than cutting or breaking off a piece of water-melon? But the comic effect was heightened by the fact that, with a serviette wrapped around his head, Charlie took on the appearance of someone suffering from toothache. Charlie thus used simple elements of actual life presented in an unexpected, and thus funny, light to achieve a comic effect.

Sensory reflection enables us to cognise a phenomenon, but not its essence, to reflect individual objects in all their vividness. The laws by which the world functions, the essence of objects and phenomena and what they have in common are things we are only able to understand by means of abstract thought, which represents a more complex form of cognition. Abstract, or rational, thought provides a deeper and fuller reflection of the world than sensory cognition. The transition from sensory cognition to abstract thought is a major leap in the process of cognition, a leap from the cognition of facts to the cognition of laws.

The main forms of abstract thought are concept, judgement and inference.

Concept is a form of thought which reflects the substantial features of one object or a class of homogeneous objects. In the context of language, concepts are expressed in words ("briefcase", "trapezium") or groups of words, i.e., phrases ("student at the school of medicine", "social worker", "River Nile", etc.). *Judgement* is a form of thought in which something is affirmed or denied about objects, their features and relations. Linguistically, judgement is expressed in terms of a narrative sentence. Judgements may be simple or complex. For example, "Locusts devastate fields" is a simple judgement, whilst "Spring has come and the rooks have arrived" is a complex judgement consisting of two simple ones. A judgement may be true of false.

Inference is a form of thought which uses one or more truthful judgements, called premises, to obtain a conclusion in accordance with certain rules. There are many forms of inference, which are the subject of the science of logic. Let us give two examples.

1) All metals are substances.
Lithium is a metal.

Litium is a substance.

The first two judgements, written above the line, are called premises, whilst the third judgement is the conclusion.

2) Plants are divided into annuals and perennials.
This plant is an annual.

This plant is not a perennial.

In the process of cognition, we seek truthful knowledge. Truth represents a correct reflection in the human consciousness of phenomena and processes taking place in nature, society and thought. The truth of knowledge is its consistency with reality. The laws of science represent truth. We can also attain truth from forms of sensory cognition, like sensation and perception. The understanding of truth as the correspondence of knowledge to things goes back to ancient philosophers, notably, Aristotle.

How do we distinguish truth from falsehood? The criterion of truth is practice. By practice, we mean the entire social and productive activity of man under certain historic conditions, i.e., people's material productive activity in the field of industry and agriculture, and also the class struggle (in class societies), the national liberation movement, political activity, the fight for peace, social revolutions, scientific experiments, etc.

The practice of man and of mankind is the test, the criterion of the objectivity of cognition. Thus, before a machine of any kind is introduced into mass production, it is tested in practice. Aircraft are checked out by test pilots, the effect of medical preparations is ascertained first on animals before, their suitability having been proven, they are used to heal human beings.

Peculiarities of abstract thought

Abstract thought is a form of *mediated and generalised reflection of reality.* Using forms of sensory cognition, we are able to take direct cognisance of things and their properties (we see that this flower is red, hear that the sea is roaring, etc.). Abstract thought enables us to obtain additional knowledge from that we already have, without resorting directly to experience or to what the sense organs indicate. For example, a doctor uses symptoms to arrive at a judgement about the nature of an illness, information from archaeological excavations leads to judgements about the life of people in previous centuries, mathematical calculations are used to adjust the trajectory of rockets, etc.

Abstract thought makes it possible to achieve cognition of the world in generalised forms, one of which is concept. For example, we form the concept "teacher" by singling out the general features common to all teachers, and other concepts are formed in a similar manner. By generalising the knowledge they have obtained, people use abstract thought to discover the laws of nature, society and cognition, penetrate the essence of phenomena and the law-governed link between them.

Thinking is the highest manifestation of consciousness. Consciousness is secondary in terms of its origin, representing a reflection of existence. But consciousness, including abstract thought, is active by nature. Having taken cognisance of objective laws, man uses them in his own interests. The active nature of thought is manifested in the fact that man makes theoretical generalisations, forms concepts and judgements and constructs inferences and hypotheses. By relying on previous knowledge, he gains the ability to make forecasts, to establish plans for the development of various branches of the economy, science, education, etc. The active nature of thought manifests itself in man's creative activity, in the ability to picture things and phenomena as displayed in scientific, artistic and other forms of imagination. Abstract thought determines the aim, means and nature of man's practical activity.

Another peculiar feature of abstract thought is thus the *active reflection of the world* and *participation in its transformation*. In practice, first and foremost productive practice, man turns the ideal into the material and carries out scientific ideas in the products of his labour.

Another specific feature of abstract thought is its indissoluble link with language. Thought is a reflection of objective reality, whilst language represents a means of expression, a way of fixing thoughts and transmitting them to other people.

The link between thought and language will be dealt with at length in § 3.

§ 2. The Concept of Logical Form and Logical Law. The Truthfulness of Thoughts and The Formal Correctness of Reasoning

Formal logic is the science dealing with the laws and forms of correct thought. Let us clarify what is meant by a logical form and a logical law.

The concept of logical form

The logical form of a concrete thought is the structure of this thought, i. e., the way in which its constituent parts are linked. Logical forms and laws are not abstract constructs, but represent a *reflection* of the objective world. However, this reflection does not constitute the entire content of the world existing outside us, but rather its structural links, which are inevitably embodied in the structure of the links in our thoughts.

The structure of a thought, that is, its logical form, can be expressed with the aid of symbols. Let us clarify the structure (logical form) of the following three judgements: "All carp are fish", "All people are mortal", "All butterflies are insects". Their content is different, but the form is one and the same: "All S are P". They include S (a subject), i. e., a concept of the object being judged, P (a predicate), i. e., a concept of a property displayed by the object, a copula ("are") and a quantifier ("all").

The following two conditional judgements have one and the same form: 1) "When iron is heated, it expands"; 2) "When a student studies logic, he raises the accuracy of his judgement". The form of these judgements is the following: "When S is P, then S is P_1".

Logical laws

The observance of the laws of logic is an essential condition for obtaining the truth in the process of reasoning. The following are usually considered the fundamental laws of formal logic: 1) law of identity; 2) law of non-contradiction; 3) the law of the excluded middle; 4) the law of sufficient reason. They will be discussed in detail in a separate chapter. These laws express the definitiveness, the non-contradiction and the provability of thought.

Logical laws act independently of people's will and are not created by their will or desire. They are a reflection of the links and relations between the things of the material world. The universal human nature of the laws of formal logic is to be found in the fact that in all historical epochs people of all classes and all nations think according to the same set of logical laws. Apart from the laws of formal logic, correct thought is also subject to the laws of materialist dialectics: the law of unity and conflict of opposites, the law of the mutual transformation of quantitative and qualitative changes, the law of negation of negation.

The truthfulness of thought and the formal correctness of reasoning

The *concept of truthfulness* or *falsehood* refers only to the concrete content of a specific judgement. If the judgement correctly reflects what happens in reality, it is true. Otherwise it is false. For example, the judgement "All wolves are predators" is true, whilst the judgement "All mushrooms are poisonous" is false.

The *concept of the formal truthfulness of reasoning* refers only to logical actions and mental operations. If our premises are true, and if we apply the laws of thought to them in a correct manner, the conclusion must accord with reality. If the premises of the inference include a false one, we may, following the rules of logic, obtain a true or a false statement. In order to show this, let us take the following inference:

All metals are solids.
Mercury is not a solid.

Mercury is not a metal.

In this inference, the conclusion turned out to be false, because the first premise represents a false judgement. For the conclusion to be true, both premises must be true judgements (provided that the rules of logic are adhered to). If the rules of logic are not observed (and the premises are true), we may also obtain a true or a false conclusion. For example:

All tigers are striped.
This animal is striped.

This animal is a tiger.

In the second inference, both premises are true judgements, but the conclusion obtained may be either true or false, since one of the rules of inference was violated.

From the viewpoint of content, thought can thus provide a true or a false reflection of the world, and from the angle of form it may be logically correct or incorrect. Truthfulness is attained when the thought accords with reality, and correct thinking depends on observance of the laws and rules of logic. It would be wrong to regard the following concepts as the same (confuse them): "Truthfulness" ("truth") and "correctness". The same applies to the concepts "falsehood" ("lie") and "incorrectness".

The theoretical and practical significance of logic

It is perfectly possible to reason logically, correctly construct inferences and disprove the points made by an opponent without knowing the rules of logic, just as people often express their thoughts in a language without studying grammar books. A knowledge of logic, however, raises the culture of thought, increases the clarity, consistency and cogency of reasoning.

It is particularly important to know the principles of logic when assimilating new knowledge, in the course of study and during preparations to deliver speeches and reports. A knowledge of logic helps to recognise logical errors in the speech and writings of others, to find more concise and correct ways of refuting these errors and to avoid making them.

Given the scientific and technological revolution and the growing flow of scientific information, particular importance attaches to the rational structuring of the teaching process. Extensive methods, calling for an expansion in the volume of the information being assimilated, are giving way to intensive ones based on a rational selection of the most essential, crucial elements from the entire flow of new information. The acquisition of the knowledge of logic makes it easier to master the rational methods of, and approaches to convincing reasoning and the development of creative thought.

Logical thinking is not an inborn quality. In order to develop it, one must familiarise oneself with the principles of logic as a science, which, in the two millennia it has existed, has accumulated theoretically substantiated and practically proven methods of, and approaches to rational reasoning and argumentation. Logic promotes the emergence of self-awareness and the intellectual development of the individual, helping him to form a scientific world outlook.

In the scientific world, in polemics, everyday life and study, we are constantly called upon to use truthful judgements to derive others and to refute false judgements and incorrectly constructed proofs. The conscious observance of the laws of logic disciplines our thought, makes for better arguments, greater efficiency and productivity, and helps to avoid errors, a factor of particular importance for students.

§ 3. Logic and Language

The forms and laws of correct thinking are the subject studied by logic. Thought is a function of the human brain. Labour enabled man to rise from the animal kingdom and was the foundation for the emergence of consciousness (including thought) and language. Thought is indissolubly associated with language. Language, according to Marx, is the immediate actuality of thought. In the course of collective labour, humans developed a need to communicate and pass on their thoughts to one another, without which the actual organisation of collective labour would be impossible.

The functions of natural language are numerous and have many aspects. Language is the means people use for everyday communication, the means of communication in scientific and practical activity. Language allows people to pass on their accumulated knowledge, their practical abilities and life's experience from one generation to another, to instruct and raise the younger generation.

Language is an information system based on signs, the product of man's intellectual activity. The accumulated information is conveyed by means of the signs (words) that go to make up language. "In language there is only the *universal*."[1] "Every word (speech) already *universalises*... The senses show reality; thought and word – the universal."[2]

Speech may be oral or written, voiced or silent (as, for example, in the case of the dumb), be external (addressed to others) or internal speech expressed in natural or artificial language. Scientific language, which is based on natural language, is used to formulate the propositions of philosophy and all other sciences – history, geography, archaeology, medicine (which, along with "living" national languages, uses the now "dead" Latin) and many other sciences.

Language is not only a means of communication but an important component part of each nation's culture. The Russian language has now become the language in which the Soviet Union's various nationalities communicate with each other, one of the most important factors in the flowering and mutual influence of these peoples' national cultures, helping them to assimilate the riches of world civilisation.

Artificial scientific languages grew up on the basis of natural languages. They include the language of mathematics, those of symbolic logic, chemistry, physics, and also algorithmic languages for computer programming, such as ALGOL 65, FORTRAN, COBOL, PL1, BASIC and others, all of which are extensively used in modern computers and computer-based systems. Programming

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[1] V.I. Lenin, *op. cit.*, p. 275.
[2] *Ibid.*, p. 272.

languages are sign systems used to describe the processes involved in solving problems on computers. In recent times, there has been a growing trend for man to "communicate" with computers in natural language, so that he can use a computer without assistance from programmers as intermediaries.

A *sign* is a material object (phenomenon, event) used to take the place of some other object, property or relation and is employed to obtain, store, process and transmit communications (information, knowledge).

Signs are divided into linguistic and non-linguistic categories. Non-linguistic signs include copies (for example, photographs, fingerprints, reproductions, etc.) or indicators (for example, smoke is a sign of fire, an excessive body temperature, of illness), signals (a bell, for example, to signify the start or the end of a lesson), symbols (such as road signs) and other kinds of signs. There is a separate science called semiotics which is the general theory of signs. One type of signs are linguistic ones used for the purpose of communication. One of the major functions of linguistic signs is to denote objects. Those used as such are called names.

A *name* is a word or a phrase denoting a certain object. (The words "denotation" and "name" are regarded as synonyms.) The term object here is understood in an extremely broad sense. It includes things, properties, relations, processes, phenomena, etc. relating to nature, the life of society, human mental activity, products of man's imagination and results of abstract thought. So a name is always the name of some object. Although objects are subject to change and flux, they do retain a certain qualitative definiteness and a relatively stable essence which is denoted by the name of the object in question.

Names are subdivided into 1) *simple* ("book", "London", "Leibnitz" and *complex*, or *descriptive*, ones ("the biggest waterfall in Canada and the United States", "a planet of the solar system", "the most northerly nuclear power station in the world"). In a simple name, there are no parts with a meaning of their own, whilst in complex names there are;

2) *proper* names, that is those of individual people, objects, events (Honoré de Balzac, the Volga) and

common ones (name of a class of objects), for example, "house", "Olympic Games participant".

Every name has a *meaning* and a *sense*. The *meaning of a name* is the object it denotes.[1]

The *sense of a name* is the way in which the name denotes a given object, i. e., information about the object contained in the name. Let us illustrate this with the help of examples. One and the same object can have a multiplicity of names (synonyms). For example, the sign "4" can be expressed as "2 + 2" or "9 − 5", both being the names of one and the same object, the number "4". The various expressions used to denote one and the same object have one and the same meaning, but a different *sense* (i. e., the senses of the expressions "4", "2 + 2" and "9 − 5" differ).

Let us give other examples to illustrate what is implied by the meaning and sense of a name. Thus, sign expressions like "the great Russian poet Alexander Pushkin (1799–1837), "the author of the narrative poem *Eugene Onegin*", "the author of the story *The Queen of Spades*", "the poet fatally wounded in a duel with d'Anthès" have one and the same meaning (they denote Pushkin the poet) but different senses.

Such expressions as "the deepest lake in the world", "the freshwater lake in Eastern Siberia at a height of about 455 metres", "the lake with over 300 tributaries and the only one source", "the lake with a depth of 1,620 metres" all have the same meaning (Lake Baikal), but different senses inasmuch as these linguistic expressions present Lake Baikal with the aid of various qualities, i. e., they give different information about the Baikal.

The correlation between the three concepts "name", "meaning" and "sense" may be schematically presented in the following way (see Fig. 1.)

This scheme is suitable not only if the name is a proper one, i. e., denotes one object (the figure 4, Pushkin, Lake Baikal), but also when it is a common

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[1] In place of the word "meaning", logical literature uses other (synonymous) terms, most frequently "denotatum", and sometimes "designatum" or "nominatum".

Fig. 1

one (for example, "man", "lake"). Then the meaning of the name will denote not one single object, but a whole class of homogeneous objects (for example, the class of lakes or the class of dogs, etc.), and the chart will continue to operate with the above qualification.

Logic distinguishes between expressions which constitute nominative functions and those which represent propositional functions. An example of the first variety is "$x^2 + 1$", "the father of y", "the difference between the numbers z and 5", whilst "x is a poet", "$7 + y = 10$" and "$x > y - 7$" are examples of the second variety. Let us examine these two types of functions.

The *nominative function* is an expression which, when variables are replaced by constants, comes to denote an object. Let us take the function "the father of y". If we replace y with the name "the writer Jules Verne", we arrive at the object "the father of the writer Jules Verne" (in this case, the name of a person).

A nominative function is an expression which in itself does not represent the name of any object and requires a certain degree of elaboration in order to become the name of an object. Thus, the expression $x^2 - 1$ does not represent any object, but if we "specify" it by replacing x with the number 3 (the figure denoting this number), we obtain the expression $3^2 - 1$, which is the name of the number 8, that is a definite object. Similarly, the expression $x^2 + y^2$ does not denote an object, but if we replace x and y with any figures chosen at random, say "4" and "1", it comes to be the name of the number 17. Such expressions as $x^2 - 1$ and $x^2 + y^2$, which require elaboration, are called functions, the former of one, and the latter of two, arguments.

The *propositional function* is the name given to an expression which contains a variable and is turned into a true or a false judgement when the variable is replaced by the name of an object from a certain object field.

The following are examples of propositional functions: "z is a town"; "x is a Soviet cosmonaut"; "y is an even number"; "$x + y = 10$"; "$x^3 - 1 = 124$".

Propositional functions are divided into one-place ones, which contain one variable and are called qualities (for example, "x is a composer", "$x - 7 = 3$", "z is a carnation") and those containing two or more variables, which are called relations (for example, "$x > y$"; "$x - z = 16$"; "the volume of cube x is equal to the volume of cube y").

Let us take as an example of a propositional function "x is an odd number". Replacing x with the number 4, we obtain "4 is an odd number", which is false. But if we choose the number 5, we then obtain "5 is an odd number", which is true.

We may illustrate this with a number of concrete examples. It is essential to state which of the functions given are nominative functions and which are propositional functions, to define the number of variables in the expressions and obtain from them names or sentences which express judgements (true or false ones).

a) "the difference between the number 100 and x". This is a one-place nominative function; thus, 100–6 is the name of an object, the name of the number 94.

b) "$x^2 + y$". This is a two-place nominative function with two variables, if we replace x with the number 5 and y with the number 7, we obtain the name of an object, namely the number 32.

c) "y is a famous general". This is a one-place propositional function. If we replace y with "Alexander Suvorov, born on November 24, 1730", we obtain a truthful judgement, namely, "Alexander Suvorov, born on November 24, 1730, is a famous general", which is expressed in the form of a narrative sentence.

d) "z is the composer who wrote the operas x and y". This is a three-place propositional function; it is turned into a false judgement if we replace z with the name "Bizet", x with "*Aida*" and y with "*La Traviata*", i.e., the judgement "Bizet is the composer who wrote the operas

Aida and *La Traviata*", expressed in the form of a narrative sentence.

The concept of a propositional function is widely used in mathematics. All the equations with one unknown which schoolchildren solve from their initial years of education, are one-place propositional functions, e.g., $x + 2 = 7$, $10 - x = 4$.

In solving an equation, we normally say that it has a set of solutions, which is the same as the expression "a set of truths to this propositional function". For example, the set of roots belonging to the equation $3x + 6 = 7 \cdot (x - 2)$, where x is a real number, is the number 5. In other words, the equation $3x + 6 = 7 \cdot (x - 2)$, is a propositional function whose set of truths is equivalent to the single number 5.

In equalities containing one or more variables are also propositional functions. For example, $x < 7$, or $x^2 - y > 0$.

Semantic categories

Expressions (words and phrases) belonging to natural language and having some kind of independent sense can be divided into what are known as *semantic categories*, which include: 1) sentences: narrative, hortative and interrogative; 2) expressions playing a certain part in the composition of a sentence; descriptive and logical terms.

The judgements are expressed in the form of narrative sentences (e.g. "London is a city", "A cow is a mammal"). In these sentences, the subjects are "London" and "cow" and the predicates are "city" and "mammal".

Descriptive terms comprise:

1. *Names of objects*, words or phrases denoting specific (material or ideal) objects or classes of homogeneous objects (e.g. "Aristotle", "7", "first cosmonaut", "ship", "interesting film", etc.).

In the judgement "Delhi is the capital of India", we find three names of objects, "Delhi", "capital" and "India". The name of the object "Delhi" plays the role of the subject whilst the names "capital" and "India" go to make up the predicate ("capital of India").

2. *Predicators* are linguistic expressions (words or phrases) denoting qualities or relations whose presence

in the relevant objects is confirmed or denied in judgements (e.g. "white", "electrically conductive", "to be a town", "less", "to be a number", "to be a planet", etc). Predicators occur in both simple and complex forms. One-place predicators denote qualities (e.g. "talented", "bitter", "large"). Complex predicators denote (express) relations. Two-place predicators include "equal", "larger", "mother", "remembers", etc. For example, "the size of plot *A* is equal to the size of plot *B*", "Maria Vasilyeva is Seryozha's mother". "Between" is an example of a three-place predicator (e.g. "the city of Moscow is located between the cities of Leningrad and Rostov-on-Don").

3. *Functional signs* are expressions which denote object functions ("ctg α", "+", "$\sqrt{\ }$", etc.).

Moreover, language also has what are known as *logical connectives* (logical constants).

Natural language has words and phrases like "and", "or", "if ... then", "equivalent", "as strong as", "not", "wrong that", "any" ("each", "every"), "some", "besides", "only", "the who", "neither ... nor", "although", "if and only if" and many others expressing logical constants.

Symbolic (or mathematical) logic normally employs such connectives in the form of conjunction, disjunction, negation, implication, equivalence, universal and existential quantifiers and a number of others.

In symbolic logic, logical connectives (logical constants) are expressed in the following way. [Hereinafter the letters *a*, *b*, *c*, etc. will be taken to signify arbitrary propositions (simple judgements) or, more precisely, they are variables for the propositions.]

Conjunction is represented by "and" and denoted by $a \wedge b$ or $a \cdot b$ or $a \& b$ (e.g. "The lectures finished (*a*) and the students went home (*b*)").

Disjunction is represented by "or" and denoted by $a \vee b$ (inclusive disjunction) and by $a \dot{\vee} b$ (exclusive disjunction). Inclusive differs from exclusive disjunction insofar as in the latter the judgement is only true if one, and not both, of its component judgements is true, whilst in the case of inclusive disjunction, both judgements may be true at the same time. "He is a chess or a soccer player" is denoted as $a \vee b$. "Now Petrov is either at home or at the institute" is denoted as $a \dot{\vee} b$.

Implication is represented by the connective "if ... then" and denoted by $a \to b$ or $a \supset b$ (e. g. "if the weather is fine, we shall go to the woods").

Equivalence is represented by the connectives "if and only if", "when and only when" and "equivalent to" and denoted by $a \equiv b$ or $a \leftrightarrow b$ or $a \rightleftarrows b$.

Negation is represented by the connectives "not" and "wrong that" and denoted by $\bar{a}$, $\neg a$ or $\sim a$ (e.g. "it's raining" (a), "it's wrong to say that it's raining" ($\bar{a}$)).

Universal quantifiers are represented by the word "every" ("any", "each", "none") and denoted by $\forall x P(x)$, (e. g. "All snakes are reptiles").

Existential quantifiers are represented by the words "some", and "exists" and denoted by $\exists x P(x)$ (e. g. "Some people have higher education").

Let us depict the various semantic categories in the form of a chart.

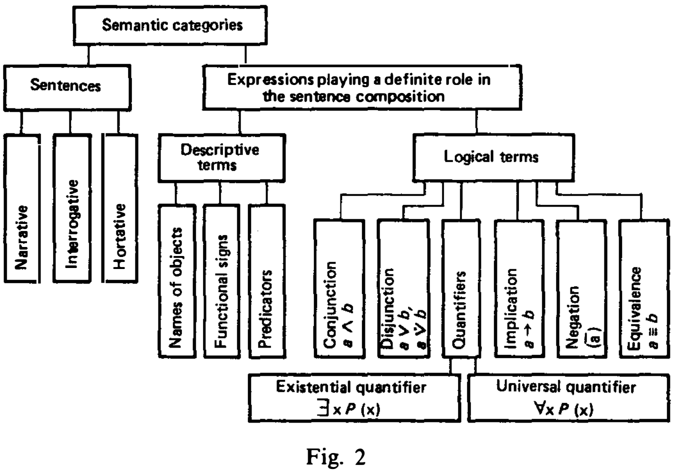


Fig. 2

Examples

1) Define the descriptive and logical terms in the judgement: "All organisms are unicellular or multicellular". In this judgement, the descriptive terms are "organism", "multicellular organism" and "unicellular organism", and the logical terms are "all" and "or".

2) Define the semantic categories to which the following expressions belong: a) leaves having fallen on the ground (a descriptive term, the name of an object); b) leaves fell on the ground (a judgement expressed in the form of a narrative sentence); c) a pushing force acts on every body immersed in liquid (a judgement expressed in the form of a narrative sentence); d) the specific gravity of copper (a descriptive term, the name of an object); e) are you going to the library today? (an interrogative sentence not containing any judgement); f) John's brother (a descriptive term, the name of an object).

Let us show how it is possible to use semantic categories to reveal the logical structure of thoughts. Below we have four complex judgements whose structures are to be expressed by way of formulas employing the logical connectives described above.

1) If I finish writing the story (*a*), translate the article (*b*) and then have some time to spare (*c*), I shall go visit my parents in the country (*d*) or take it easy in the city (*e*).

The formula is $(a \wedge b \wedge c) \rightarrow (d \dot{\vee} e)$.

Here, the letter *a* denotes the judgement, "I finish writing the story"; the letter *b* the judgement "I translate the article"; the letter *c* the judgement, "I have some time to spare"; the letter *d*, "I shall go visit my parents in the country"; and the letter *e*, "I shall take it easy in the city".

2) "If a man has not had the nerves to control himself from childhood or youth, they will not get used to irritation and will obey him". The formula is: $(\bar{a} \wedge \bar{b}) \rightarrow$ $\rightarrow (\bar{c} \wedge d)$.

Here, the letter *a* denotes the judgement, "A man has not had the nerves to control himself from childhood". And since we have a negation ("not had"), we write $\bar{a}$.

3) Even virtue can become a vice when it is incorrectly applied.

In order to reveal the structure of this judgement, we must first clearly define the cause and effect, and for this purpose the judgement in question had to be put into a clearly defined logical form, "If virtue is incorrectly applied (*a*), it can become a vice (*b*).

The formula is $a \rightarrow b$.

4) “If a child has grown a rose in order to admire its beauty, if his only reward for this work was the enjoyment of beauty and the creation of this beauty for the happiness and joy of another person, then he is incapable of evil, meanness, cynicism and insincerity.”

The formula is $(a \wedge (b \wedge c \wedge d)) \rightarrow (\bar{e} \wedge \bar{f} \wedge \bar{m} \wedge \bar{n})$

Exercises

I. Give the object (denotatum) and semantic (conceptual) meaning of the expressions “cosmonaut”, “metal”, “the author of the novel *Le Rouge et Le Noir*”, “seaman”, “participant in the international forum, ‘For a Nuclear-Free World, for the Survival of Mankind’ ”.

II. Say which of the following expressions are nominative functions and which are propositional (one, two or three-place) and obtain from them names or sentences expressing judgements (true or false).

1. Mr. x has red hair.
2. Product of the numbers 7 and y.
3. Sum of $x^2 + y^2$.
4. Z is the capital of a modern state.
5. Writer x is a contemporary of writer y.
6. The number x is greater than seven.
7. The natural number x is greater than seven.
8. River x flows into the Caspian Sea.
9. x multiplied by 9.
10. y can be divided by 5 with a remainder.
11. $x + y > 10$.
12. $x^2 - y^2 = (x + y) \cdot (x - y)$.
13. x and y are sisters.
14. x is the grandfather of y.
15. Town x is located between towns y and z.
16. $2x^3 - 1 + y^2$.
17. The outstanding scientists x and y, who lived in the 19th century.

III. Identify the semantic categories to which the following expressions belong:

a) the dog barks;
b) a loudly barking dog;
c) the highest mountain peak in the world;
d) a song ringing out in the still of night;
e) the song rang out in the still of night;

f) a performer of traditional songs;
g) some geometric figures are flat;
h) automated control system.

IV. Express the following complex judgements in symbolic form:

1. If you rise at dawn and go into the garden or the park, you can hear wonderful bird songs.

2. If the given rectangle is a rhombus, its diagonals are perpendicular to each other and divide its angles into equal halves.

3. To see injustice and remain silent is to be a party to it oneself.

4. If you love children, are full of a thirst for knowledge, have a kind heart and dream of dedicating yourself to interesting creative work, then you can safely choose the profession of a teacher.

5. If Peter were to pass by the workers, he would lose no time in lending a hand, helping either to mow the grass, cut down a tree or chop the wood.

6. If this figure is a square, then its sides are equal as are its diagonals, and all its angles are right angles, and it is a regular polygon, which means that one can inscribe a circle in it and describe a circle around it.

7. The paths along which the children pass from one building to another are kept clean, and if they are soaked in rain during bad weather, the pupils' feet only become covered in water, but not in mud and dust.

8. If a child has put part of his heart and soul into working for other people and found personal joy in this work, he cannot become an evil, unkind person.

Chapter II
CONCEPTS

§ 1. Concept As a Form of Thought

A *concept* is the reflection of concrete objects and their properties using forms of sensory cognition – sensation, perception and representation. For example, we sense the properties of an orange as being its round shape, its orange colour, its sweet taste and its aroma. The sum of these and other properties forms the perception (concrete image of a single object) of the orange in question. A concept reflects merely the substantive features of objects.

Features are the ways in which objects resemble and differ from each other. Objects may be identical in terms of their properties (e. g. sugar and honey are both sweet), or may differ in their properties (e. g. honey is sweet, but wormwood is bitter).

Features may be *essential* or *inessential*. A concept reflects the sum of essential features, i. e., those which, taken individually, are necessary, and together are sufficient to distinguish the object in question from all others.

Features may be *distinguishing* or *non-distinguishing*. There are features which apply to just one object and allow it to be distinguished from similar objects. For example, the individual distinguishing feature of the planet Mercury is that its orbit is closer to the sun (than those of all other planets in the solar system). The individual distinguishing feature of Yuri Gagarin is that he was the world's first cosmonaut. There are also common distinguishing features attributable to many homogeneous objects.

The *distinguishing features* of any class of objects are those which are attributable only to objects belonging to the given class. For example, the distinguishing features of a human being are the following: an ability to create means of production, an ability to think in abstract terms and the presence of speech.

Non-distinguishing features are those which belong not only to the objects in question. For example, the non-distinguishing features of metals are their thermal and electrical conductivity; for lions, their non-distinguishing features are the fact that they belong to the group of predatory animals, to the vertebrates, etc.

As may be seen from the examples, distinguishing and non-distinguishing features may be constituted not only by individual features but by their totality.

Thus, *conception* is a form of thought reflecting the substantive features of a single object or a class of objects.

Linguistic forms of expressing concepts may be words or phrases (groups of words). For example, "book", "jungle", "racing car", "strong earthquake", "bright spring sun". There are homonyms which have a different meaning and express different concepts but sound the same (e.g. the Russian word "mir" (world, peace) may express either an objective reality or the absence of war, "kosa" in Russian has three different meanings, etc.). There are also synonyms, which have the same meaning, i.e., express one and the same concept, but a different sound (e.g. car – automobile, aubergine – egg plant, kindergarten – nursery).

The main logical approaches to the formation of concepts are analysis, synthesis, comparison, abstraction and generalisation.

A concept is formed by generalising the substantive features (i.e., properties and relations) attributable to a series of homogeneous objects.

In order to single out the substantive features, it is essential to abstract oneself (isolate oneself) from the non-substantive, of which any object has a very large number. This purpose is served by comparing objects, by setting them against each other. To identify a series of features, it is necessary to carry out an analysis, i.e., to split up in one's thoughts the object in question into its component parts, elements, aspects, individual features, and then to undertake the opposite operation – synthesis (amalgamation in one's thoughts) of the parts of the object, its individual features, including the distinguishing features, into a single whole.

Analysis as an approach employed in the formation of concepts is often preceded by a practical analysis, i. e., the splitting, the dissection of an object into its component parts. A synthesis in terms of thought is preceded by a collection of the parts of an object to form a single whole, taking into account the need to correctly position the parts during assembly.

Analysis is the division of objects into their component parts and the identification of their features.

Synthesis is the combination of the component parts of an object, or its features, obtained in the course of analysis, to form a single whole.

Comparison is the establishment of similarity or dissimilarity between objects in terms of their substantive and insubstantive features.

Abstraction is the selection of certain properties of an object and their isolation from others. The task involved is frequently to single out the substantive properties of an object and isolate them from those which are insubstantive and of secondary importance.

Generalisation is the amalgamation of individual objects in one or another concept.

The logical approaches listed above are used in the formation of new concepts, both in a scientific context and in learning.

§ 2. The Intension (Content) and Extension of Concepts

Any concept has an intension and an extension. The *intension (content) of a concept* is the sum of the basic essential features of the object or class of homogeneous objects reflected in the concept. The content of the concept rhombus is the sum of two substantive features, "being a parallelogram" and "having equal sides".

The *extension of a concept* is the class of objects generalised in it. In objective terms, i. e., outside human consciousness, there exist varying objects, such as animals, for example. The extension of the concept "animal" refers to the set of all animals which exist at the present day, existed in the past and will exist in future.

The class (or set) is made up of individual objects, which are called its *elements*. Depending on their number, the set may be finite or infinite. For example, the set of planets in the solar system is finite, whereas the set of natural numbers is infinite. Set (class) A may be called a subset (subclass) of set (class) B if every element of A is an element of B. This relationship between subset A and set B is called the inclusion of subset A in set B and denoted as $A \subset B$. For example, the relationship between species and genus (e. g. the class "fir" is included in the class "tree").

The relationship between element a and class A can be expressed as follows: $a \in A$, where a is Victoria and A is a lake.

Classes A and B are identical (coincidental) if

$$A \subset B \text{ and } B \subset A, \text{ which is denoted as } A \equiv B.$$

The law of inverse proportion between the extension and intension of a concept

The extension of one concept may be included in the extension of another and represent just part of it. For example, the concept "motor boat" is fully included in the extension of another broader concept called "boat" (it is a part of the concept "boat"). The intension of the first concept is broader and richer (contains more features) than the intension of the second. By generalising this kind of example, we may formulate the following law: the broader the extension of a concept, the narrower its intension, and vice versa. This is called the law of inverse proportion between the extension and intension of a concept. The law of inverse proportion between extension and intension indicates that the less the information on objects included in the concept, the broader the class of objects and the less defined its composition (e. g. "plant"), and vice versa; the greater the information contained in the concept (e. g. "edible plant" or "edible cereal plant"), the more narrow and more closely defined the set of objects. This law refers to concepts related to each other as genera and species.

§ 3. Types of Concepts

Concepts can be classified in terms of extension and intension. In terms of extension, they may be divided into particular and general concepts.

The extension of a *particular* concept is one element (e. g. "the great American writer Theodore Dreiser", "the River Amazon", "the capital of the Soviet Union", etc.). The extension of a *universal* concept includes a number of elements, more than one (e. g. "car", "briefcase", "state", etc.).

Among general concepts, we distinguish concepts with an extent equivalent to a *universal* class, i. e., a class which includes all objects considered in the given field of knowledge or all those within the confines of the reasoning in question (these concepts are known as universal). For example, real numbers in arithmetics; plants in botany; constructive objects in constructive mathematics.

Apart from general and particular concepts, in terms of extension there also exist *empty concepts (with zero extension)*, i. e., those whose extension is an empty set (e. g. "perpetual motion machine", "man who lived 300 years", "Santa Claus", figures from nursery rhymes, fables, etc.).

In terms of intension, we may distinguish between the four following pairs of concepts.

Concrete and abstract concepts

Concrete concepts are those which reflect individual objects or classes of objects (both material and ideal). They include concepts like "house", "witness", "novel", "Alexander of Macedon", "earthquake", etc.

Abstract concepts are those which refer not to an entire object, but to any of its features taken in isolation from it (e. g. "whiteness", "injustice", "honesty"). In reality white clothes and unjust actions do exist, as do honest people, but "whiteness" and "injustice" do not exist by themselves and cannot be perceived directly, but only as qualities of specific objects. Apart from individual properties of objects, abstract concepts may reflect relations between them (e. g. inequality, similarity, identity, resemblance, etc.).

Relative and non-relative concepts

Relative concepts are those referring to objects where the existence of one presupposes the existence of another ("children" – "parents", "pupil" – "teacher", "high" – "low", "north magnetic pole" – "south magnetic pole").

Non-relative concepts refer to objects which exist independently of any external object ("table", "man", "blast furnace", "village").

Positive and negative concepts

Positive concepts are used to describe the presence of one or another quality or relation in an object. For example, a literate person, greed, backward pupil, good deed, exploiter, etc.[1])

If a word contains the particle "non" and is not used without it (e. g. nonchalance), then the concepts expressed by such words are also regarded as positive. The English language knows no word like "chalance". So that "non" in this case does not fulfil the function of negation. Thus, the concept "nonchalance" is positive since it denotes the presence of a certain quality in the object, even though the quality in itself may be negative.

Negative concepts are those which indicate the absence of a certain quality, (e. g. "illiterate person", "nonentity"). Linguistically, these concepts are expressed by a word or a phrase containing a negative particle, such as "non" or "un", etc. This particle is attached to the relevant positive concept and *fulfils the function of*

[1] In logic, the concepts "greed" and "exploiter" are positive since they refer to a certain feature attributable to an object (in this case, a human being) – "to be an exploiter", "to be greedy". The logical nature of a concept does not always coincide with the assessment of objects reflected in the concept (e. g. economic, moral and other values). It goes without saying that exploiters and greedy people do not evoke a positive assessment, but an extremely negative one. The concept "natural calamity" is regarded as positive in logic, although in actual practice a natural calamity is considered a negative, undesirable phenomenon which brings people grief, destruction and suffering.

negation. The positive concept (*A*) and the negative one (*not-A*) are contradictory.

Collective and non-collective concepts

Collective concepts are those which regard a group of homogeneous objects as a single whole (e.g. "regiment", "herd", "shoal", "constellation"). The content of a collective concept cannot be attributed to each of the elements which make up the extent of the concept in question. For example, we cannot say that one tree is a forest, that one ship is a fleet or one soccer player is a team. Collective concepts may be general (e.g. "grove", "choir") or individual (e.g. "The Great Bear constellation", "the Lenin Library in Moscow", "the first crew of an orbital station).

The content of a *non-collective* concept may be attributed to each separate object in a given class as belonging to it ("pen", "river", "toy", "plant"). This will lead to true judgements. For example, we may say of each plant that it is indeed a plant, and this judgement is true.

In judgements (statements), general and particular concepts may be used in either a non-collective (partitive) or a collective sense. Let us take the judgement, "All apples in this basket are ripe". In it, the concept "apple in this basket" is general and used in a non-collective sense, i.e., each individual apple is ripe. In the judgement "All apples in this basket weigh five kilogrammes", the concept "apple in this basket" is used in a collective sense, i.e., they weigh five kilogrammes when taken all together, and not each separately.

For the purpose of clarification, let us give the following examples.

Give the logical characteristics of the concepts "collective", "carelessness" and "poem".

Collective is general, concrete, non-relative, positive and collective.

Carelessness is general, abstract, non-relative, negative and non-collective.

Poem is general, concrete, non-relative, positive and non-collective.

§ 4. Relationships Between Concepts

The objects in the world around us are interconnected and have a bearing on each other. For this reason, certain relationships also exist between the concepts which reflect these objects. A link between two objects may be extremely distant in terms of content. It may be the simple fact that these two concepts reflect certain objects or properties of objects in the real world (e.g. "irresponsibility" and "thread", "novel" and "brick"). These kinds of concepts, which are very far away from each other in terms of their content and have no common features, are called *non-comparable*, whilst all others are referred to as *comparable*.

Comparable concepts are divided by their extension into *compatible* (whose extensions coincide in full or in part) and *incompatible* (whose extensions do not coincide even in one element).

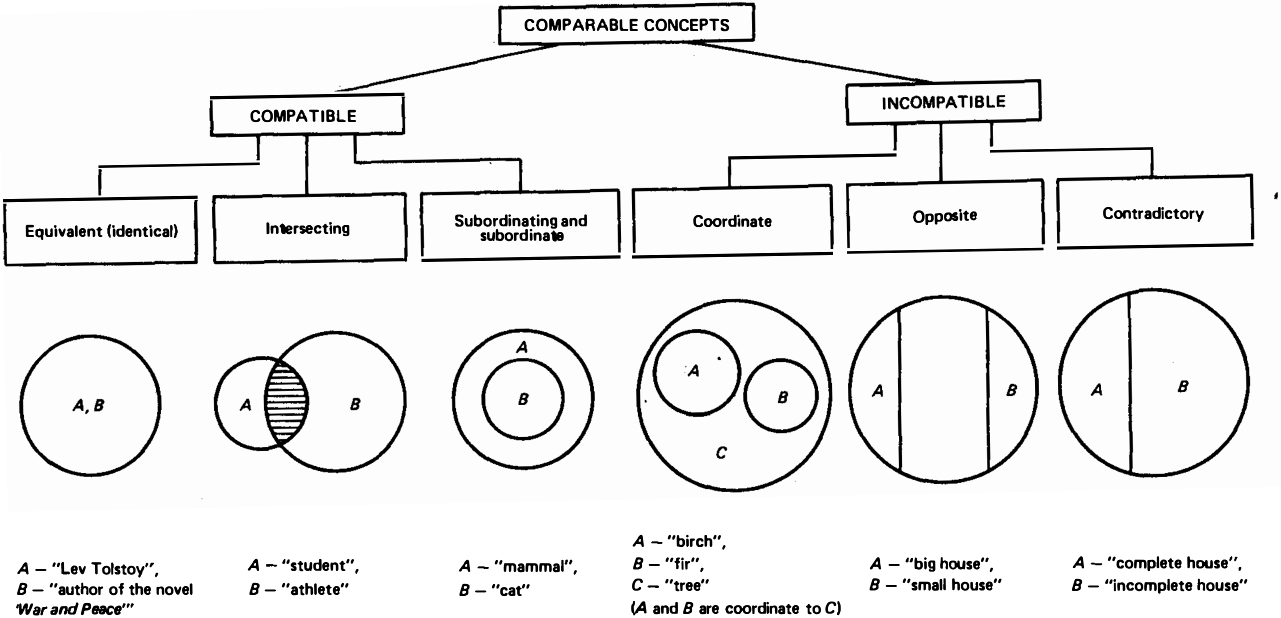
Types of compatibility: equivalence (identity), intersection, subordination (genus–species relationship)

Relations between concepts are expressed with the aid of circular diagrams (Euler's[1] circles), where each circle denotes the extension of a concept. Even if the concept is particular, it is still denoted by a circle.

Equivalent or *identical* concepts differ in terms of their intension but coincide in their extension, i. e., they refer either to one and the same class, consisting of one element, or one and the same class of objects, consisting of more than one element. Examples of equivalent concepts are: 1) "Volga"; "longest river in Europe"; 2) "Anton Chekhov"; "author of the play *The Cherry Orchard*"; 3) "equilateral quadrangle"; "square"; "equiangular rhombus". The extensions of identical concepts are depicted by superposed circles.

Concepts whose extents coincide in part, i. e., have common elements, are said to *intersect*. The following

[1] Leonhard Euler (1707–1783), an outstanding mathematician, physicist and astronomer.



COMPARABLE CONCEPTS
COMPATIBLE
INCOMPATIBLE
Equivalent (identical)
Intersecting
Subordinating and subordinate
Coordinate
Opposite
Contradictory
A, B
A
B
A
B
A
B
C
A
B
A
B
A — "Lev Tolstoy",
B — "author of the novel 'War and Peace'"
A — "student",
B — "athlete"
A — "mammal",
B — "cat"
A — "birch",
B — "fir",
C — "tree"
(A and B are coordinate to C)
A — "big house",
B — "small house"
A — "complete house",
B — "incomplete house"

Fig. 3

pairs are examples of this: "engineer" and "woman"; "school-pupil" and "stamp collector"; "athlete" and "student". They are depicted by intersecting circles (see Fig. 3). The shaded section of the two circles refers to students who are also athletes or (what is the same thing) athletes who are also students, whilst the left-hand section of circle *A* refers to students who are not athletes and the right-hand section of circle *B* refers to athletes who are not students.

The *subordination* relationship is said to obtain when the extension of one concept is fully included (contained) in the extension of another concept, but does not exhaust it. It is the relationship between a genus and a species; *A* is the superordinate concept ("mammal") and *B* the subordinate concept ("cat").

Types of incompatibility:
coordination, opposition, contradiction

Coordination is the relationship between extents of two or several concepts which are mutually exclusive, but belong to a certain, more general generic concept (e. g. "fir", "birch" and "pine" all belong to the extension of the concept "tree"). They are depicted by separate circles which do not intersect within a larger circle. They are species belonging to one and the same genus.

Opposite (contrary) concepts are those which are species of one and the same genus, but where one of them contains certain features and the other not only does not have these features but replaces them with others (i. e., opposite features). The words used to express opposite concepts are *antonyms*. Examples of opposite concepts are: "bravery" – "cowardice", "white" – "black". The extensions of these pairs of concepts are separated by the extension of a third concept, including, for example, "green".

Two concepts are said to be *contradictory* when they are species of one and the same genus, but one concept refers to some feature which the other negates, rules out, without replacing it with other features. If we denote one concept as *A* (e. g. "complete building"), then the other object which is in contradiction to it should be denoted as *non-A* (i. e., "incomplete building"). Euler's circles, when used to express the extension of such concepts, are split into two parts (*A* and *non-A*), with no third concept between them. For example, paper may be white or not white; a human being may be honest or not honest; an animal may be a mammal or a non-mammal, etc. Concept *A* is positive and concept *non-A* is negative.

Concepts *A* and *non-A* are antonyms.

Examples. Determine the relationships between the following concepts and depict them using Euler's circles.

1. House, incomplete house, stone house, building.
2. Athlete, worker, poet.

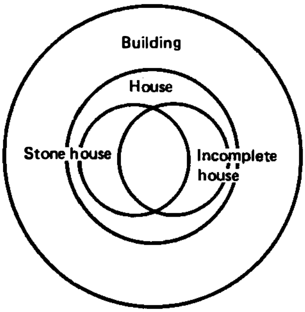


Fig. 4

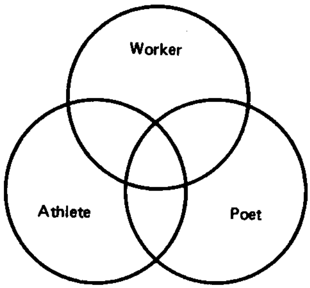


Fig. 5

§ 5. Definition of Concepts

The *definition* of a concept is a logical operation which reveals the content of the concept or establishes the meaning of a term.

In *defining* concepts, we clearly indicate the essense of the objects reflected in the concept, reveal the content of the concept and thus distinguish between the range of defined objects and all other objects. Thus, in defining the concept "trapezium", we distinguish it from other quadrangles, from a rectangle and a rhombus, for example. "A trapezium is a quadrangle in which two sides are parallel and the other two are not" (1). Let us give some other types of concept definitions belonging to two different types of definition. "Substances whose solutions conduct electric current are called electrolytes" (2). "Flora is the name given to the totality of plant species growing on a certain territory" (3). "Subtraction is the act of finding one of the components by reference to the sum and the other component" (4).

The concept whose content is to be revealed is called the *definiendum* (*Dfd* for short) whilst the concept used to define it is called the *definiens* (*Dfn* for short).[1]

Real and nominal definitions

If we are defining a concept, then the definition will be *real*. If, on the other hand, we are defining a term denoting a concept, then the definition will be *nominal*. Of the definitions above, (1) and (4) are real, whilst (2) and (3) are nominal.

Nominal definitions are also used to introduce new terms, short names to replace more complex object descriptions. For example, "A habit is an act in which the individual operations have become automatic as a result of a repetition".

We also use nominal definitions to introduce signs as replacements for terms. For example, "conjunction is denoted by $\wedge$ or &", "C is the speed of light", "The tangent of angle α is denoted by $\mathrm{tg}\,\alpha$", etc.

It is typical of nominal definitions that they contain the words "is called". Nominal definitions are often found in books on mathematics. For example, the following nominal definitions will be encountered in a geometry course, "A cone is called circular if its base is a circle" or "A circular cone is called a cone of revolution".

Definitions may be explicit or implicit. Explicit definitions are those in which *Dfd* and *Dfn* are given and some relation of equality and equivalence established between them. The most widespread type of explicit definition is definition by means of the closest genus and the specific distinction. It is used to establish the substantive features of the object being described.

Examples: 1. "A regular polygon is one in which all sides and all angles are equal."

2. "A barometer is an instrument for measuring atmospheric pressure."

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[1] For further details about definition see D. P. Gorsky, *Definition*, Progress Publishers, Moscow, 1981.

3. "Grotesque is one of the means for the satirical depiction of life, distinguished by excessive exaggeration and a combination of the real and the fantastic."

A feature indicating the range of objects from which the set for definition is to be singled out is called a *generic feature* or a *genus*. The generic features in the examples given above are the concepts "polygon", "instrument" and "means for the satirical depiction of life".

Features used to distinguish the set to be defined from the number of objects, falling under a generic concept are called *specific distinctions*. One or more of them may be used to define the concept of specific features (distinctions).

Explicit definitions of concepts also include *genetic definitions*. A genetic definition is a definition of an object by means of the way in which it, and no other, is formed (it represents the said object's specific distinction). Genetic definition is a variety of definition using genus and a specific distinction.

Let us give some examples of genetic definitions from the fields of mathematics and physics. 1. A circular cone can be obtained by revolving a right-angled triangle around one of its catheti. 2. A sphere is a geometric body obtained by revolving a semicircle (or a circle) around its diameter. 3. Acids are complex substances obtained from acidic residues and atoms of hydrogen capable of being replaced with atoms of metals or changing places with them. 4. The corrosion of metals is a reduction-oxidation process resulting from the oxidation of metal atoms and their transformation into ions.

The use of concept definitions in the teaching process

Definition through genus and specific distinction as well as nominal definition are techniques widely used in the teaching process. Let us give a series of examples. The following may be regarded as definition by means of the nearest genus and the specific distinction: "Higher nervous activity is the sum of numerous interlinked nervous processes taking place in the cortex"; "Heredity is the term used to describe the general property of all organisms in preserving and transmitting structures and functions received from their ancestors to their descendants". Books on inorganic chemistry contain numerous *nominal definitions* of concepts, for example: "A taking up by chemical or physical forces of the molecules of gas or dissolved substances or of liquids by the surfaces of solids or liquids with which they are in contact is called adsorption". In physics textbooks we find less definitions using a genus and specific distinction and more nominal ones, for example: "The temperature at which a substance melts is called the substance's melting point". Nominal definitions may be applied to the following concepts in physics: "heat transfer", "solidification (or crystallisation) temperature", "specific melting heat", "evaporation", "condensation", "boiling point", "specific evaporation heat", "current strength", "electrical force" and many others. There are also real definitions to be found in this context. In works on geography, the opposite is the case, with preference being given to real definitions by means of genus and specific distinction. For example, "A mineral is a natural body which is homogeneous in terms of chemical composition and physical properties". Books on mathematics, history, literature and other subjects contain numerous definitions. The definition of concepts is one of the most widespread means of passing on information in a concentrated form.

Lecturers who are mastering the methodology of putting over their subject must first and foremost organise their work with *fundamental, support* concepts and laws, and be able to select the most essential information in the teaching process. Clear identification of fundamental concepts promotes a rise in the theoretical level of teaching.

In studying history, literature and other humanities, students familiarise themselves with the fundamental and support concepts and not only assess features of these concepts, but also organically link their content with the present time. For example, a precise understanding of the concept "culture" calls for a profound study of the achievements of material culture. In accordance with the two types of production – material and intellectual – it is accepted practice to divide culture into material and intellectual forms, which means that students need to have a clear understanding of the content of the concepts "material culture" and "intellectual culture" and form a more general concept of "culture" on this basis. Even today there exist many varied definitions of the concept "culture" in academic literature.

Students are faced with the following overall tasks: achieving a profound understanding of the fundamental concepts in the course being studied, working out an integral system to reveal the most important concepts of the chosen discipline, gradually broadening their scope and adding to their structure. This is the way in which support concepts are mastered.

The rules of explicit definition.
Possible errors in definition

1. *A definition should be proportionate*, i. e., the extent of the defining concept should be equal to the extent of the concept to be defined. $Dfd \equiv Dfn$. This rule is often violated, giving rise to logical errors in the definition. Types of logical errors include:

a) A *broad definition* is said to obtain when $Dfd < Dfn$. This error is to be found in the following definitions: "Gravitation is the interaction of two material bodies". "A horse is a mammal and a vertebrate". (This definition makes it impossible to distinguish the concept "horse" from the concepts "cow" and "goat".) The concept "circle" may be incorrectly defined as follows: "It is a figure described by the moving end of a segment when its other end is fixed, or a figure formed by the moving end of a compass". Using this definition, it is impossible to distinguish the concept "circle" from the concept "curve", since it is not indicated that a circle is a *closed* curve.

b) *Narrow definition* occurs when $Dfd > Dfn$. For example, "Conscience is man's awareness of his responsibility to himself for his actions" (and to society?). "Productive forces is the name given to instruments of labour and also to people with their skills and modes of labour". (Productive forces include all means of production and not just instruments of labour.)

c) *A definition which is broad in one respect and narrow in another.* In these incorrect definitions, we find both

Dfd > *Dfn* and *Dfd* < *Dfn* (in different relations). For example, "A barrel is a vessel for storing liquids". From one viewpoint, it is a broad definition, since a teapot, a bucket, etc. may also be used to store liquids. From another viewpoint, the definition is narrow, since a barrel is suitable for the storage of solid bodies and not only liquids. A similar error is contained in the following definition of the concept "teacher": "A teacher is a person who instructs children".

2. *A definition must not form a vicious circle.* This is the case when *Dfd is defined by means of Dfn* and *Dfn* was previously defined by *Dfd*. In the definition, "Rotation is a movement around the axis", a vicious circle will obtain if the concept "axis" was defined with reference to the concept "rotation" ("an axis is a straight line around which rotation takes place").

A vicious circle is also formed when the concept to be defined is described in terms of itself, just using different words, or when the concept being defined forms part of the defining concept. Such definitions are called tautologic.

In exposing the logical error made by the bourgeois ideologue Struve, who incorrectly defined "economy" Lenin wrote: "Economy is defined as economic management. A statement of the obvious..."[1]

The following definitions are tautologic: "Negligence is when a person takes a negligent attitude to his duties"; "Quantity is a characterisation of an object from the quantitative point of view".

This use of tautologies in thought (and in speech) is logically incorrect, e. g. cooperative cooperation, hard difficulty, progressing progress, draw a drawing, etc. Sometimes we encounter such expressions as "The law is the law" and "A fact is a fact", etc., which are means of emphasis. The predicate does not give any information about the subject, since the two are identical.

3. *A definition should be precise and clear.* This rule means that the meaning and extent of the concepts contained in *Dfn* should be clear and well defined. Definitions of concepts should be free of ambiguity; it is

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[1] V. I. Lenin, "Socialism Demolished Again", *Collected Works*, Vol. 20, 1977, p. 198.

inadmissible to substitute metaphors, comparisons, etc. for them.

The following judgements do not represent definitions: "Architecture is music congealed", "The lion is the king of the animal world", "The camel is the ship of the desert", "Tact is the reason of the heart", "Ingratitude is a kind of weakness".

Implicit definitions

In contrast to explicit definitions, which have the structure $Dfd \equiv Dfn$, in implicit definitions Dfn is replaced by the context, a set of axioms or a description of the means by which the object being defined is built.

Contextual definition makes it possible to clarify the meaning of an unknown word expressing a concept with reference to the context, without resorting to a dictionary for a translation, if the text is in a foreign language, or to a defining dictionary, if the text is in one's native language.

The values of unknown quantities in equations are given in implicit form. If we have a simple equation, e. g. $10 - y = 3$, or a quadratic one, e.g. $x^2 - 7x + 12 = 0$, then by solving them and finding the values of the roots of these equations, we give an explicit definition to $y (y = 7)$ and to x ($x_1 = 4$ and $x_2 = 3$).

Inductive definitions are those in which the term to be defined is used to express the concept assigned to it as its sense. An example of an inductive definition is the definition of the concept "natural number" using the actual term "natural number".

1. 1 is a real number.
2. If n is a real number, then $n + 1$ is a real number.
3. There are no other real numbers other than those indicated in 1 and 2.

This inductive definition leads us to the set of natural numbers: 1, 2, 3, 4 ... This is the algorithm used to build the set of natural numbers.

Axiomatic definition

What is known as the axiomatic method is widely employed in modern mathematics and in mathematical logic. The Soviet mathematician P. S. Novikov gives the following example. Let there be a system of any elements (denoted by x, y, z...) and a relationship established between them which is expressed by the term "precedes"; we can make the following statements in this regard (i. e. the following two axioms);

1. No object precedes itself.
2. If x precedes y and y precedes z, then x precedes z.

These two axioms have thus been used to define the system of objects of the type "x precedes y". For example, let x and y be people and the relationship between x and y be "x is older than y". Then statements 1 and 2 come into effect. If objects x, y and z are real numbers and the relationship "x precedes y" is equivalent to "x is less than y", then statements 1 and 2 also operate. Statements (i. e., axioms) 1 and 2 *define* the systems of objects with a single relationship.

Methods resembling the definition of concepts

It is impossible to give definitions of all concepts (nor is it necessary), so science and teaching also employ other methods of introducing concepts. These methods are similar to definition and include description, characterisation, clarification by means of an example, etc.

Description is the listing of the external features of an object with a view to its non-rigorous distinction from similar objects. Description yields a sensory-visual image of the object by activating the recipient's creative or reproductive imagination. Description includes both substantive and non-substantive features.

Description is widely used in various genres of fiction (e. g. Lev Tolstoy's description of the heroine Anna Karenina from his novel of the same name, Stefan Zweig's description of the appearance of Honoré de Balzac, his father and other people, descriptions of landscapes, trees, birds, etc.), in historical literature (description of the appearance of military commanders, monarchs, diplomats, etc.) and as descriptions of the external appearance of machines and appliances, etc.

Here is an example. Alexandre Dumas gives the following description of the young sailor Dantès:

He was youth of 18 to 20, tall, slender, with beautiful black eyes and jet-black hair; his entire appearance breathed the calmness and determination that is peculiar to people who have become accustomed to fighting danger since childhood (Alexandre Dumas, *Le Comte de Monte Cristo*, Tome 1–2, Paris, Bureau de L'Écho des Feuilletons, 1846, p. 2).

When criminals are being sought, descriptions are issued of their appearance, notably their distinctive marks, so that people can recognise them and report on their whereabouts.

Characterisation is the listing of just some of the inner properties of a person, phenomenon or object, and not its external appearance, as is the case with description.

In his reminiscences of Lenin, Gorky presents a vivid characterisation of the revolutionary leader that is positively outstanding for its depth of thought and literary form:

It is difficult to draw his portrait... He was simple and straight, like everything that he said.

His heroism was almost completely devoid of any outward shine; his heroism was not a rare one in Russia – the modest ascetic dedication of a Russian revolutionary intellectual convinced of the possibility of social justice on earth, the heroism of a man who renounced all earthly joys for the sake of hard work in the name of people's happiness... This was a wise and perspicacious man...[1]

Sometimes characterisation occurs by way of reference to one feature. For example, Marx called Aristotle, "the greatest thinker of antiquity ...".[2]

A characterisation of literary heroes is achieved by listing their practical qualities, their moral and socio-political views, their traits of character and temperament and the aims they set themselves. The characterisation of these figures makes it possible to clearly and accurately identify the typical features of the artistic image in question.

We frequently encounter a *combination of description and characterisation.*

Students come across a combination of description and characterisation in the study of biology, geography,

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[1] Maxim Gorky, *Works*, Vol. 23, Moscow, p. 419 (in Russian).

[2] K. Marx, *Capital*, Vol. 1, p. 384.

history, chemistry and other disciplines. For example, "petroleum is an oily liquid lighter than water with a dark colour and a sharp smell. The most important property of petroleum is its combustibility. When burnt, it yields more heat than coal. Petroleum deposits are found deep in the bowels of the earth."

This technique is also used widely in fiction. In his short story *The Man who Lived in a Shell*, Anton Chekhov gives a description of the outward appearance and a characterisation of Belikov, the man in the case.

> Belikov was famous for never stirring out of his house, even in the best weather, without an umbrella, galoshes and a wadded coat. His umbrella he kept in a case, he had a case of grey suède for his watch, and when he took out his pen-knife to sharpen his pencil, he had to draw it out of a case, too; even his face seemed to have a case of its own, since it was always hidden in his turned-up coat-collar. He wore dark glasses, and a thick jersey, and stopped up his ears with cotton wool, and when he engaged a droshky, made the driver put up the hood. In fact, he betrayed a perpetual, irrepressible urge to create a covering for himself, as it were a case, to isolate him and protect him against external influences. Reality irritated and alarmed him and kept him in constant terror, and, perhaps to justify his timidity, the disgust which the present aroused in him, he always praised the past and things which had never had any existence. Even the dead languages he taught were merely galoshes and umbrellas between himself and real life...
>
> Belikov tried to keep his thoughts in a case, too. Only those circulars and newspaper articles in which something was prohibited were comprehensible to him... In his eyes permission and indulgence always seemed to contain some doubtful element, something left unsaid, vague. If a dramatic society or a reading-room or a café were allowed to be opened, he would shake his head and say gently: "It's a very fine thing no doubt, but ... let's hope no evil will come of it." (Anton Chekhov, *Selected Works*, Raduga Publishers, Moscow, 1984, pp. 88–89.)

Chekhov goes on to describe what an absurd insignificant figure is this man in a case and how he has nonetheless contrived to hold the whole town in a state of fright with his obsessive fear of life.

> Teachers were all afraid of him. Even the headmaster was. Just think! Our teachers are on the whole a decent, intelligent set, ... and yet this mite of man, with his eternal umbrella and galoshes, managed to keep the whole school under his thumb for fifteen years! And not only the school, but the entire town! (*Ibid*)

Where fear reigns, worthlessness gains the upper hand, and Chekhov wanted people to understand this grim logic, this dialectic of fear.

Clarification by means of an example is employed when it is easier to give one or more examples to illustrate the concept in question than to strictly define it by reference to the genus and specific distinction. The concept "desert animals" is explained by listing the species inhabiting the desert: camel, antelope, gazelle, tortoise, lizard, etc.

The concept "mineral" is explained by listing its varieties, i.e., by giving examples: oil, coal, granite, etc.

A variation on this approach is an *ostensive* definition, which is often employed in foreign-language teaching. An object, or a drawing of it, is displayed and then its name is given. This method is also used sometimes to clarify the meaning of unknown words in the native language.

Another method of replacing definition is *comparison*.

The famous Soviet educationalist Vasily Sukhomlinsky compared a child's brain to a rose bloom.

> We teachers are concerned with the most tender, the most fragile, most delicate thing in nature, a child's brain. Picturing a child's brain, I imagine a delicate rose bloom on which a drop of dew is trembling. What care and delicacy is required in order to pick the flower without destroying the drop. We need the same care every minute: after all, we are concerned with the most fragile and delicate thing in nature, with the thinking matter of a growing organism (Sukhomlinsky, V.A., *On Education*, 1975, p. 87, in Russian).

In science, comparison makes it possible to reveal similarities and distinctions of the objects to which it refers. For example, "The body of a jellyfish is wobbly and resembles an umbrella"; "Kidneys are small paired organs having the shape of beans"; "A pea flower resembles a butterfly with its wings closed"; "The ovaries in the pestil of a sweetbrier are concealed in a swollen receptacle resembling a gablet". In all the examples given, the common denominator (basis of comparison) is the shape.

Comparison, when it is the artistic depiction of reality, makes it possible to identify general or similar features in two objects and to use imagery to express this similarity in a vivid form.

Lermontov wrote, "Love is a fire which dies away if it is not fed" (M. Y. Lermontov, *Works* in four volumes, Moscow, Leningrad, Vol. 4, p. 415, in Russian). Shakespeare made an interesting comparison in one of his sonnets:

Mark how one string, sweet husband the another,
Strikes each in each by mutual ordering;
Resembling sire and child and happy mother,
Who, all in one, one pleasing note do sing:
Whose speechless song, being many, seeming one,
Sings this to thee: "Though single wilt prove none.

(William Shakespeare, *The Sonnets of W. Shakespeare*, London, Kegan Paul, French & Co., 1881, p. 116).

Artistic comparisons frequently include the words "like", "as if", etc.

Differentiation is a method allowing the object in question to be separated from similar objects. For example, "Hysteria is not an illness, but a character trait. Its most important feature is autosuggestibility".

The meaning of definitions in science and reasoning

Apart from taking into account the demands of formal logic when defining a concept, one also needs to consider the methodological requirements made on a definition. A definition of a concept may be formulated following a comprehensive study of the object to which it refers, and although we never fully achieve it, comprehensiveness protects us from errors and denaturalisation; it is essential to study the object not in a static state but in a dynamic one, in the process of its development. One needs to take account of the criterion of practice and the concreteness of truth. Investigation is a concrete analysis of a concrete situation. Lenin's definition of the concepts "matter", "class", "sensation" and others were the result of scientific generalisation. Lenin warned against confusing concepts and pointed out the inadmissibility of using diffuse and vague formulations. All scientific terminology is built up in line with methodological demands, and logic is called upon to assist scientists representing the individual disciplines to systematise scientific terms.

Applied in conjunction with concrete knowledge, the methodological demands on the definition of concepts and the rules of definition dictated by formal logic enable us to arrive at more precise definitions of the concepts used in social, natural and engineering sciences as well as in everyday practice.

The precise definition of concepts, and the correct revelation of their content and extent, are important not only in establishing scientific terminology, but also in specifying commonplace statements.

The role of concept definitions in science results from the fact that definitions, since they express our knowledge about the objects in the world around us, are a substantive element in the cognition of the world. Definitions are given to all fundamental concepts in every discipline.

In the legal sciences, major practical importance accords to the precise definition of such concepts as "bribe", "speculation", "lawful defence", "crime", "legal responsibility" and many others.

As for the role of concept definition, it should be pointed out that one cannot demand more from a definition of a concept than it is able to provide.

§ 6. Division of Concepts. Classification

Division is a logical act by means of which the extent of the concept (set) to be differentiated is divided into a number of subsets using the chosen basis for division. For example, syllables are divided into stressed and unstressed; the sense organs are divided into organs of sight, hearing, smell, touch and taste. Whilst definition is used to reveal the intension of a concept, division is employed to demonstrate its extension.

The feature according to which the extension of a concept is divided up is known as the *basis of division*. The subsets into which the extension of the concept is divided are called *elements of division*. The concept divided is a genus and the elements of differentiation are species within that genus coordinated amongst each other, i. e., they do not intersect (have no common elements). Let us give some examples of the division of concepts. Depending on the source of energy, power stations are divided into hydroelectric, helioelectric, geothermal, wind-driven, and thermoelectric power stations (a nuclear power station is a variety of the thermal type).

The extent of a concept may be divided up according to different criteria, depending on the purpose of division and the practical tasks involved. But every division should just have one basis at some level or other. For example, depending on their location, muscles are divided into those of the head, the neck, the trunk, the upper extremities and the lower extremities. Muscles may also be differentiated according to their shape and function. In terms of shape, they are divided into wide, long, short and circular. As regards their function, we distinguish between bending and unbending.

Rules of Concept Division

In order to achieve correct division, it is essential to observe the following rules.

1. *Proportionality of division: the extent of the concept being divided should be equal to the sum of the elements of division.* For example, higher plants are divided into grasses, bushes and trees. Electric current is differentiated according to whether it is direct or alternating. If the elements of division runs into the tens or even hundreds, in order to observe the proportionality rule, we may use the words, "and so on", "and others" or similar ones. For example, we may answer the question, "What nations and nationalities live in the USSR?" as follows, "Russians, Ukrainians, Byelorussians, Georgians, Armenians, Tadjiks, Uzbeks, Lithuanians, Estonians, Yakuts, Nenets, Lapps, Chechens and others".

The violation of this rule leads to two kinds of error:

a) *incomplete division*, when not all members of the genus are listed. The following divisions are erroneous: "Energy is divided into mechanical and chemical forms" (this does not refer, for example, to electrical or atomic energy). "Arithmetical operations are divided into addition, subtraction, multiplication, division and involution" ("root extraction" is missing);

b) *division with surplus elements*. An example of this kind of erroneous division is, "Chemical elements are divided into metals, non-metals and alloys". Here the surplus element is "alloys", since the sum of "metals" and "non-metals" covers the full extension of the concept "chemical elements".

2. *Division must have only one basis.* This means that division cannot be based on more than one distinguishing feature.

If this rule is violated, the extension of the concepts yielded by division will overlap. The following divisions are correct, "Waves are divided into longitudinal and transversal types", "In industry steel is obtained in three ways: in oxygen converter, open-hearth and electric furnaces". The following division is incorrect, "Transport is divided into land, water, air, private and public transport", since the error has been made of substituting the basis, i. e., the division is carried out according to more than one criteria. To start with, the chosen criterion is the medium in which transportation takes place, but then the designation of the transport is made to serve as the basis of differentiation.

3. *The elements of division must be mutually exclusive,* i. e., they should not have any common elements, and be coordinate concepts whose extensions do not overlap.

This rule is closely related to the previous one, since if division takes place using more than one basis, the elements of division will not be mutually exclusive. Examples of erroneous division are, "Fractions may be decimals, proper, improper, repeating or non-repeating"; "Wars may be just, unjust, wars of liberation, predatory or world wars"; "Triangles may be right-angled, obtuse-angled, acute-angled, isosceles or similar". In these examples, the elements of division are not mutually exclusive. This is the result of mistakenly introducing various bases of division.

4. *Division should be uninterrupted, i. e., there should be no empty spaces left by the division.*

We will be making an error if we divide fertilizers into organic, nitrogenous, phosphatic and potassic. It would be correct to first distinguish between organic and mineral fertilizers and then divide mineral fertilizers into nitrogenous, phosphatic and potassic.

Types of division according to the species-forming feature and dichotomous division

When a concept is divided according to a variable feature, the basis of division is the feature according to which the specific concepts are formed; this feature is the species-forming one. For example, according to the size of an angle, angles are divided into right angles, acute and obtuse. Nuclear explosions may be divided into air, ground, underwater or underground ones (depending on the type of medium in which the explosion took place). Depending on their scale, maps are divided into large-scale, medium-scale and small-scale.

Dichotomous (twin-element) division

The extension of the concept to be divided is split into two opposing concepts: *A* and *not-A*. Examples: "Organisms are divided into unicellular and multicellular (i. e., not unicellular)"; "Substances are divided into organic and inorganic ones"; "Radioactivity is divided into natural and artificial (i. e., not natural) types"; "Societies are divided into class and classless types".

Sometimes, the concept *not-A* is again divided into two opposing concepts *B* and *not-B*, and *not-B* is then divided into *C* and *not-C*, etc.

Examples of dichotomous division.

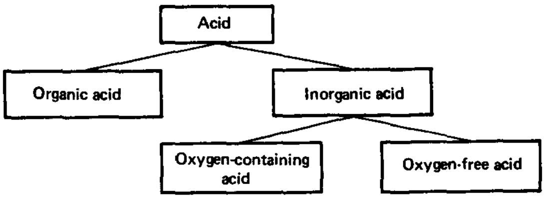


Fig. 6

Dichotomous division is convenient for the following reasons: it is always proportional; the elements of division are mutually exclusive, since every object in the set being divided belongs to class *A* or *not-A*; division takes place on just one basis. For this reason, dichotomous division is extremely widespread. It would, however, be wrong to assume that it is universally applicable in all instances.

The operation of dividing a concept is employed when it is necessary to establish the specific entities which go to make up a generic concept. Concept division should be distinguished from the division of a whole into parts. For example, "A house is divided into rooms, corridors, a roof and a porch". Parts of the whole are not distinctive elements of a genus, i. e., the concept being divided. We cannot say, "A room is a house", but we can say "A room is a part of a house".

Wide use is made of the method of breaking down a whole into its parts. The concept "human skeleton" can accurately illustrate the method of breaking down the whole into parts.

> In the human skeleton we distinguish between sections: *skeleton of the head, the trunk and the extremities*. The skeleton of the trunk consists of the spine and the thorax. *The skeleton of the extremities* consists of that of the free extremities and the skeleton of the girdle. The pectoral girdle includes the twin bones of the shoulder blades and the clavicles... the skeleton of the free upper extremities consists of the shoulder bone, the bones of the forearm and the hand. The pelvic girdle consists of the paired pelvic bones and the sacrum. The skeleton of the free lower extremities consists of the thigh, the shin and the foot.

Concepts like "skeleton of the head", "skeleton of the spine", "skeleton of the shin", "foot", are also divided into parts.

Examples of the mental splitting of a whole into parts from the field of botany: "The structure of a rye flower is made up of the flower scales, the stamens, the pistil stigma and the ovary".

The division of a whole into parts is also used in mathematics. For example, "The development of the surface of any regular prism is a flat figure made up of collateral sides in the form of rectangles and two bases in the form of polygons".

Classification

Classification is the division of objects into groups (classes) where each class occupies a fixed, defined place. Classification is extremely long-lived when it has a scientific nature. For example, the classification of fundamental particles is constantly being amended and supplemented, so that it now extends to over 200 different types. Classification differs from conventional division to the extent that it is relatively stable.

When carrying out classificatory operations, it is essential to observe all the rules applying to the division of concepts. Classification is a kind of consecutive division; it forms a developed system, where each element (species) is divided into subspecies, etc.

Classification may take place according to a species-forming feature or be dichotomous. The following are examples of classification according to a species-forming feature:

a) Mirrors are classified as plane and spherical; spherical mirrors are classified as concave and convex.

b) The classification of the concept "fruit" looks like this:

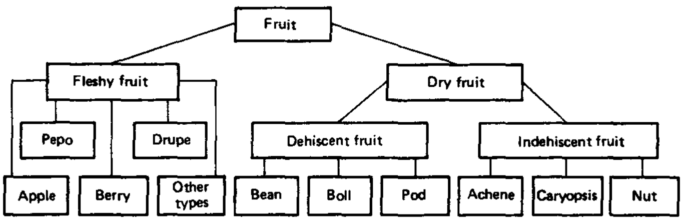


Fig. 7

Here we see a combination of two types of classification: that according to a species-forming feature and the dichotomous form.

An extremely important component of classification is the selection of the *basis of classification*, since this gives rise to various classifications of one and the same concept.

For example, "reflex": 1) *in terms of their origin* all reflexes are divided into unconditioned (inborn) and conditioned; 2) *in terms of their biological significance for the organism*, reflexes are divided into defensive, nutritive, sexual, orientating, and locomotor (i. e., reflexes concerned with the body's movement in space). 3) *In terms of the position of receptors*, reflexes are divided into exteroceptive (i. e., emerging in response to the stimulation of receptors on the surface of the body), proprioceptive (arising from stimulation of the receptors of the muscles, tendons and joints and interoceptive (aroused by stimulation of the receptors in the internal organs). 4) Reflexes may be classified *in terms of the organs which respond to stimuli.* We thus refer to respiratory, cardiovascular and other reflexes. 5) *In terms of reactions,* reflexes are classified as secretory (i. e., expressed in the secretion of liquids by glands), trophic (i. e., associated with metabolism), and motor, characterised by the contraction of transversal striated and smooth muscles.

Sometimes reflexes are classified according to what parts of the central nervous system are involved in their execution, but this classification is relative and therefore omitted. So we see how interesting and varied the classification of reflexes can be.

Classification may be effected according to essential properties (natural) or according to non-essential properties (ancillary).

Natural classification is the division of objects into groups (classes) on the basis of their essential features. If we know the group to which an object belongs, we are able to arrive at judgements about its properties. An example of natural classification is the Periodic Table of Chemical Elements devised by Mendeleyev. By classifying chemical elements according to their atomic weight, he revealed the laws governing their properties.

The fundamental principle on which Mendeleyev's Periodic Table of Elements is constructed is the classification of all elements according to groups and periods. Each group is then divided into a main and an auxiliary subgroup. Each subgroup contains elements which display similar properties. The period is the name given to the sum of elements, beginning with an alkaline metal and ending with an inert gas (with the exception of the first period).

The natural classification of animals covers up to 1.5 million species, and the classification of plants includes about 500,000 different species.

From a dialectical viewpoint, it is sometimes impossible to establish strict dividing lines, since everything is in a state of development, flux, etc. Every classification is relative and approximate, reflecting the links between the classified objects in a simplified form. There are also transitional forms which it is difficult to include in any one particular group. Sometimes this transitional group forms a group (species) of its own. For example, in the classification of the sciences, we encounter such transitional forms as biochemistry, geochemistry, physical chemistry, space medicine, astrophysics, etc.

The use of natural classifications in science and teaching

People encounter classification in the course of studying any discipline. Let us analyse several natural classifications present in natural language in which a distinction is made between *parts of speech*: independent, auxilliary and interjections. Independent parts of speech are nouns, adjectives, numerals, verbs, adverbs, and pronouns. Auxiliary parts of speech are prepositions, conjunctions, particles and modal words. Interjections form a separate group. The borders between the individual classes of words are extremely mobile, various transitional instances are observed in the study of the individual parts of speech. In describing the peculiarities of natural classification, we referred, among other things, to the existence of transitional (intermediate) species in the objects classified. A good way of vividly displaying what is involved in classification is to use tree-like graphs (or simply "trees"). The above classification of the concept "fruit" was given in the form of a tree-like graph.

The branches of Soviet law may be classified as follows: constitutional (state) law, administrative, labour, collective farm, land, civil, family, criminal law. The classification of the freedoms guaranteed to Soviet citizens might look like this: freedom of speech, freedom of assembly, freedom of street processions, freedom of demonstrations.

The following are examples of natural classifications: the classification of socioeconomic formations, types of social revolution; the classification of zones of vegetation, animals' protective markings, blood types; the classification of climatic zones; a geochronological table of eras (Cainozoic, Mesozoic, etc.) and the periods in each era; the classification of natural zones (tundra, taiga, wooded steppe, etc.); the classification of literary trends in the late 19th and early 20th centuries; the classification of numeration systems; the classification of inequalities, types of plane figures, spherical bodies (in mathematics) and many more besides.

No science can manage without relevant classifications. In this connection, both scholars and students need to know the general rules whose observance helps to avoid errors in concrete classifications.

Artificial classification is the division of objects into groups (classes) on the basis of their non-essential properties. It is used to facilitate the location of an object (or term). Artificial classification does not allow any judgements to be made as to the properties of objects (for example, a list of surnames in alphabetical order, an alphabetic catalogue of books, articles in periodicals). Subject or name indexes and also catalogues of medicines listed in alphabetical order, are examples of artificial classifications. An artificial classification is employed when compiling a list of the brightest stars in alphabetical order. Any subject index represents an artificial classification.

§ 7. Limitation and Generalisation of Concepts

Let us suppose that we know someone is a scientist and wish to find out more about him. To elaborate, we may say he is a Soviet scientist, an outstanding Soviet scientist, the physiologist Ivan Pavlov.

The logical operation performed here is one of limiting a concept. Let us give another example. We have the concept "place". By limiting it, we may obtain the concepts "city", "capital", "capital of a socialist state", "capital of the Hungarian People's Republic".

We can see that the process of limitation involves the transition from a concept with a larger extension to one with a smaller extent, i. e., from the genus to a species and from the species to a subspecies. New features are added, making it possible to narrow down the extension of the concept in question.

Limitation is a logical operation involving a transition from a generic concept to a specific one by adding species-forming features to the said generic concept.

Limitation cannot go any further than the unique concept. In the example given above this was the concept "capital of the Hungarian People's Republic."

Generalisation, the opposite operation to limitation, is the transition from a species concept to a generic one, i. e., from a concept with a smaller to one with a greater extension. This operation is effected by discarding the distinguishing feature (features). For example, by generalising the concept "Siamese cat", we arrive at the following concepts, "domestic cat", "cat", "mammal", "vertebrate", "animal", "organism".

Generalisation is a logical operation involving a transition from a specific concept to a generic one by discarding the species-forming feature (features) from the intension of the specific concept in question.

The limit of generalisation is the category.

Categories in philosophy are the most universal, fundamental concepts, reflecting the most substantive natural links and relations between actual reality and cognition.

The categories employed in dialectical materialism comprise matter and motion, space and time, consciousness, reflection, truth, identity and contradiction, content and form, quantity and quality, necessity and chance, cause and effect, etc.

Every science has its categories and uses categories taken from philosophy as well as general scientific categories (e. g. information, symmetry, etc.). In scientific cognition, we single out those categories which define the subject of an actual science (e. g. species and organism in biology).

Here is an example. Generalise and limit the concept "university Komsomol organisation".

Limitation:

1. Komsomol organisation of a university in Moscow (USSR).
2. Komsomol organisation of Moscow State University.

Generalisation:

1. Komsomol organisation of a higher educational institution.
2. Komsomol organisation of an educational establishment.
3. Komsomol organisation.
4. Mass organisation

Let us give several more examples to illustrate generalisation and limitation. Generalise and limit the concept "bear."

Generalisation:

1. Animal of the canine family.
2. Vertebrate.
3. Animal.
4. Organism.
5. Body

Limitation:

1. Brown bear.
2. Brown bear native to Europe.
3. European, brown trained bear.
4. European brown trained bear in the Soviet circus in Kiev by the name Clumsy.

Let us carry out a generalisation of the concepts "camel" and "sable".

Camel–hardiest and least demanding domesticated desert animal; hardy and undemanding domesticated desert animal; domesticated desert animal; domesticated animal; animal.

Sable–valuable fur-bearing animal; fur-bearing animal; animal.

Let us carry out a limitation of the concepts "bird" and "agricultural crop".

1) Steppe bird, rare steppe bird, rare steppe bird about one metre in height (bustard).

2) Ancient agricultural crop, ancient Russian agricultural crop, ancient Russian fibrous agricultural crop, flax.

In processes of generalisation and limitation, a distinction should be made between transitions from genus to species and relations of the whole to a part (and vice versa). For example, it is incorrect to generalise the concept "town centre" to "town" or to limit the concept "factory" to "shop", since both examples involve not a relationship between a genus and a species, but between a whole and a part.

The logical operation of generalising a concept is applied in literally all cases where definitions are given according to genus and a specific distinction. For example, "A noun is a part of speech", "Sodium is a chemical element", or, better still (by agency of the nearest genus) "Sodium is a metal".

A student of chemistry may carry out generalisation and limitation operations with the concept "acid" in the following manner. *Generalisation*: "complex chemical substance", "chemical substance". *Limitation*: "inorganic acid", "inorganic non-oxygenous acid", "HCl".

§ 8. Operations with Classes (Extensions of Concepts)

Operations with classes are the logical acts which lead us to the formation of a new (generally) class.

Possible operations with classes comprise combination, intersection, subtraction and complementation.

Unification ("addition") of classes

The unification (or *sum*) of two classes is the class of elements which belong to at least one of both classes.[1]

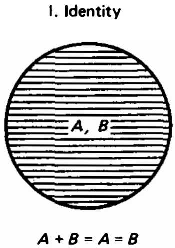


Fig. 8

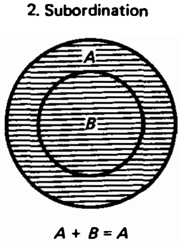


Fig. 9

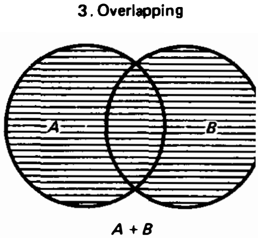


Fig. 10

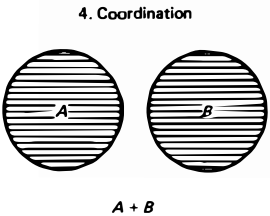


Fig. 11

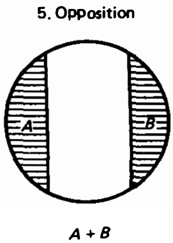


Fig. 12

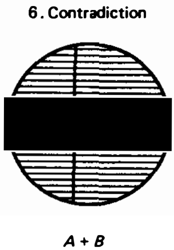


Fig. 13

[1] We could have immediately defined the combination operation for several classes, but it can always be reduced to several operations with two classes.

Unification is denoted as $A + B$ or $A \cup B$. The unification of the class of even numbers with the class of odd numbers gives the class of whole numbers. By unifying the class of poets with the class of modern poets, we obtain the class of poets. Unification can take the following six forms (Figs. 8–13).

Let us give a number of examples. If two concepts are identical, e. g. "quadrangle" (A) and "figure with four sides" (B), by unifying the two concepts $A + B$ we obtain the same concept "quadrangle" (A) or "figure with four sides (B), so that $A + B = A = B$ (Fig. 8). The unification of two classes when one is subordinate to the other can be illustrated by unifying the class "bear" (A) and the class "white bear" (B), whereupon we obtain the class "bear" (A).

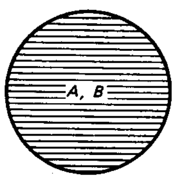
By unifying the class "plant" (A) and the class "edible" (B), we obtain a class consisting of all plants and all edible materials, including edible plants.

Intersection ("multiplication") of classes

The common part (or *intersection*) of two classes is that class of elements which are found in both sets in question, i. e., the set (class) of elements common to both sets.[1] Intersection is denoted by $A \cdot B$ or $A \cap B$; $\emptyset$ is an empty set.

For example, the operation of intersection of the classes "pupil" (A), and "football player" (B) is that of

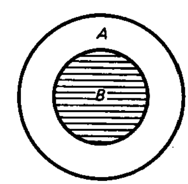
1. Identity



$A \cdot B = A = B$

Fig. 14

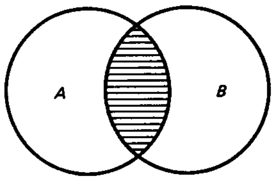
2. Subordination



$A \cdot B = B$

Fig. 15

3. Overlapping

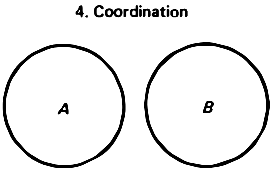


$A \cdot B$

Fig. 16

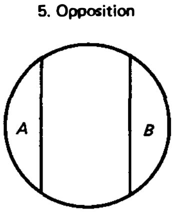
[1] Similarly to the operation of unifying classes, the intersection operation could have been immediately defined for several classes.

finding the people who are pupils and football players at the same time. It is illustrated in Fig. 16, where the common section of classes A and B is shaded.



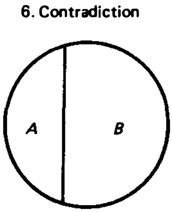
$A \cdot B = \phi$

Fig. 17



$A \cdot B = \phi$

Fig. 18



$A \cdot B = \phi$

Fig. 19

Fundamental laws of the logic of classes. Laws governing unification and intersection operations

1. Laws of idempotency.

$$A + A = A$$

$$A \cdot A = A.$$

Such laws do not exist in an elementary algebra course. In logic, the first of these laws is used to denote the following. If to the class "house", we add the class "house", we also obtain the class "house", i. e., there are not twice as many houses and the extension of the concept "house" remains the same.

2. Laws of commutativity. These laws exist in algebra, arithmetic, in the theory of sets and the logic of classes.

$$A + B = B + A$$

$$A \cdot B = B \cdot A$$

If to the class "plant" we add the class "animal", we obtain the class "organism"; we obtain the same class if we add the class "plant" to the class "animal".

3. Laws of association. These apply in arithmetic, algebra, the theory of sets and the logic of classes.

$$(A + B) + C = A + (B + C)$$

$$(A \cdot B) \cdot C = A \cdot (B \cdot C)$$

4. Laws of distributivity.

$$(A + B)\cdot C = (A\cdot C) + (B\cdot C)$$
$$(A\cdot B) + C = (A + C)\cdot(B + C)$$

5. Laws of absorption. These laws exist neither in arithmetic nor in elementary algebra.

$$A + (A\cdot B) = A$$
$$A\cdot(A + B) = A$$

These laws may be proved to operate by graphic methods.

The two laws of absorption for the "addition" and "multiplication" of classes are graphically illustrated in Figs. 20 and 21.

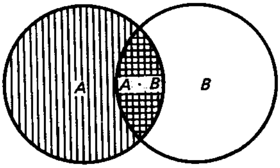


Fig. 20

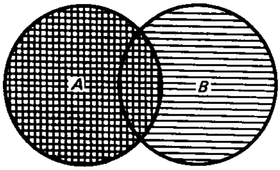


Fig. 21

The first law of absorption is depicted in Fig. 20. We need to prove that $A + (A\cdot B) = A$. To begin with, we obtain the intersection of A and B, i. e., that which is denoted in brackets as $(A\cdot B)$ and is horizontally shaded on the sketch. We then add A to $(A\cdot B)$ and obtain A. So we have A now on the left-hand side of the equation and A on the right-hand side. This means that we have graphically proved the first law of absorption.

As for the second law of absorption, $A\cdot(A + B)$, let us start by carrying out the operation in brackets, i. e., $A + B$, which is depicted in Fig. 21 with horizontal shading, i. e., we have shaded both circles. If we then locate the intersection with class A, we obtain A on the left-hand side of the equation. And A also stands on the right. We have thus proved the second law of absorption.

Let us now prove the first law of distributivity: $(A + B) \cdot C = (A \cdot C) + (B \cdot C)$.

Let us take three Euler's circles on the left and right to illustrate three concepts A, B and C (Figs. 22 and 23).

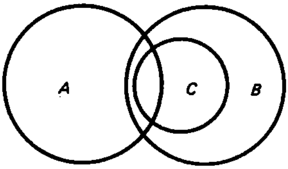


Fig. 22

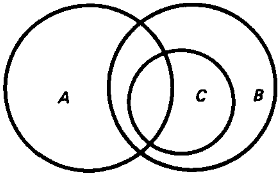


Fig. 23

The left-hand side of the equation, $(A + B) \cdot C$ is illustrated in the figure on the left (22), so that the figure on the right illustrates the right-hand side of the equation (23).

On the left, $A + B$ gives us

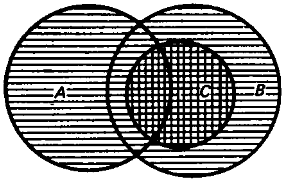


Fig. 24

The intersection of $A + B$ and C gives us C.

On the right-hand side, the intersection $A \cdot C$ gives us the small shaded section.

The intersection $B \cdot C$ gives us C.

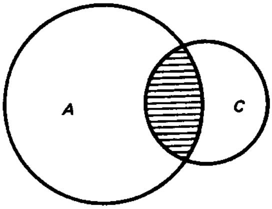


Fig. 25

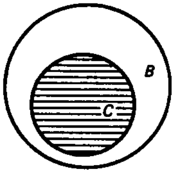


Fig. 26

The third operation is addition. By adding part of class *C* and the whole of class *C*, we obtain class *C*. So we now have *C* on both the left and the right sides of the equation.

We have thus graphically proven the first law of distributivity.

Subtraction of classes

Let us examine two sets (classes) *A* and *B*, in which *B* may not be a part of *A*. The *difference in sets* (classes) *A* and *B* is the set of elements in class *A* which are not elements of class *B*. The difference is denoted as *A* – *B*.

We may encounter the following five instances (providing that classes A and B are not empty and not universal).

First instance (Fig. 27). Class *A* includes class *B*. The difference A – B is therefore the shaded section of *A*, i.e., the set of elements which are not *B*. For example, if from the set of sounds in natural language (*A*) we subtract the set of vowels (B), we obtain the set of consonants, which is illustrated on the sketch as a shaded ring.

Second instance (Fig. 28). The difference between two overlapping classes will be the shaded section *A*. For example, the difference between the set "worker" (*A*) and the set "inventor" (B) will be the set of workers who are not inventors.

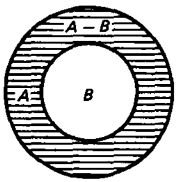


Fig. 27

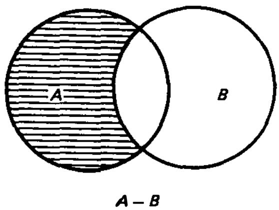
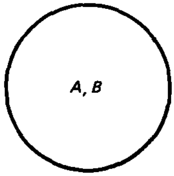


Fig. 28

Third instance (Fig. 29). If class *A* is completely included in class *B* and class *B* is completely included in class *A*, then these two classes (sets) are equal (identical).

The difference A – B will then be an empty, or null, class, i. e., a class without any elements. For example, if from the class "pine" we subtract the class "pine", the difference A – B will be equal to an empty class.

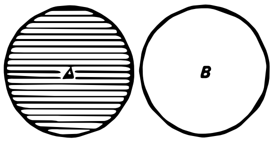
Fourth instance (Fig. 30). Class *A* and class *B* have no common elements. Then the difference $A - B = A$, since any element of class *A* is not an element of class *B*. For example, the difference between the class "table" (*A*) and the class "chair" (*B*) is equal to the class "table" (*A*).



$A - B = \phi$

Fig. 29

Coordination



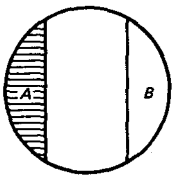
$A - B = A$

Fig. 30

By "subtracting" the classes corresponding to concepts which are in a state of opposition ["low building" (*A*), "high building" (*B*)] or contradiction ["animate object" (*A*), "inanimate object" (*B*)], the difference *A* – B also becomes equal to *A* (Figs. 31, 32).

Fifth instance (Fig. 33). If the extension of class *A* is less than the extension of class *B*, we obtain an empty class as a result of subtraction, since there are no elements of *A* which are not also elements of *B*. For example, the difference between the class "personal pronoun" (*A*) and "pronoun" (*B*) gives us an empty class.

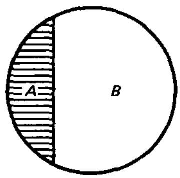
Opposition



$A - B = A$

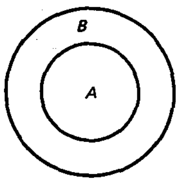
Fig. 31

Contradiction



$A - B = A$

Fig. 32



$A - B = \phi$

Fig. 33

The following laws apply to the operations involving the subtraction of classes:

1. $A - B \leqslant A$.
2. $A \leqslant B \rightleftarrows A - B = \emptyset$.
3. $A = (A \cdot B) + (A - B)$.
4. $B \cdot (A - B) = \emptyset$.
5. $B \leqslant B - (A - B)$.

In the interpretation of the algebras of logic by means of classes the expression $A \leqslant B$ denotes the inclusion of class A into class B; $A \rightleftarrows B$ denotes *equivalence* (A is then and only then if B).

Let us explain these laws.

The first law $A - B \leqslant A$ reads: "The difference between the extensions of classes A and B is less than or equal to the extension of class A". This can be seen from Fig. 29 where the difference between extensions is equal to a null class, i.e., is less than the extension of A. The other illustrations also show that the difference $A - B$ is either equal to $\emptyset$ (Figs. 29 and 33), or less than A (Figs. 27 and 28), or equal to A (Figs. 31 and 32).

The formula of the second law, $A \leqslant B \leftrightarrows A - B = \emptyset$ can be read as follows: "The extension of class A is less than or equal to the extension of class B then and only then if the difference between the extensions of classes A and B equals a null class". This is illustrated by Figs. 29 and 33.

The third law, $A = (A \cdot B) + (A - B)$ can be read thus: "The unification of the intersection of classes A and B with their difference is equal to class A". This can be seen from Figs. 27–33. For example, Fig. 27 shows the difference $A - B$ as a shaded ring. The intersection $A \cdot B = B$. The unification of the shaded ring and B gives A.

In Fig. 31 the difference $A - B = A$. The intersection $A \cdot B$ is empty. Adding a null class to A we obtain class A. On the left is also class A.

The other cases are illustrated similarly.

The fourth law, $B \cdot (A - B) = \emptyset$ reads like this: "The intersection of the difference $A - B$ with class B gives a null class". Fig. 27 shows that between $A - B$ (the shaded ring) and B there are no common elements, i.e., we obtain an empty set. Fig. 32 shows that $A - B = A$.

The intersection $A \cdot B = \emptyset$.

The fifth and last law of subtraction, $B \leqslant B - (A - B)$ reads: "The extension of class B is less than or equal to the difference between the extensions of class B and $(A - B)$".

The five formulas we have considered characterise subtractive structures, i. e., those in which one of the operations is the subtraction of classes.

Complement of class A

Class A' which, added to class A, gives the object field under consideration (we shall denote it as 1) and, intersecting with class A gives $\emptyset$, i. e., for which $A + A' = 1$ and $A \cdot A' = \emptyset$, is called *the complement of class A*. Following from the above, $A' = 1 - A$, therefore the complementation of class A may be regarded as a particular case of subtraction (from a universal class denoted as 1). Subtracting the class of even numbers (A) from the class of integers (1) we obtain the class of odd numbers (A'), since any integer is either an even or an odd number and there are no even numbers that are odd. Thus, the shaded part of Fig. 34 denotes the complement of A, i. e., A'.

The laws of complementation.

Complementation has the following laws: $1^1 = \emptyset$; $\emptyset^1 = 1$; $(A^1)^1 = A$.

Operations with classes (extensions of concepts) are interconnected with the entire complex of logical operations with concepts. Thus we express, with the help of Euler's circles, the relations between different concepts: "student" (A), "athlete" (B) and "skier" (C) as well as, using different shadings, operations $C - B$ and $A \cdot B \cdot C$.

The use of operations with classes in dealing with concepts.

Operations with classes (extensions of concepts) are used in conjunction with the entire complex of logical actions applied to concepts. In the process of dichotomous division and dichotomous classification, the laws applying to the complementation operation, $A + A' = 1$ and $A \cdot A' = \emptyset$ act in a most direct manner. For example, the universal, denoted by 1 and consisting of "vertebrates" is divided into A ("mammals") and A' ("non-mammals"), so that the intersection of A and A' is an empty set. In the case of division and classification according to the species-forming feature, the unification ("sum") of species, i. e., elements of division, should yield the entire universal. For example, if there are four elements of division, so that $A + B + C + D = 1$ and the elements of division must not intersect but be coordinated concepts. Their intersection must form an empty set, i. e., $A \cdot B = \emptyset$, $A \cdot C = \emptyset$, $A \cdot D = \emptyset$, $A \cdot B \cdot C = \emptyset$, $A \cdot C \cdot D = \emptyset$, $B \cdot C \cdot D = \emptyset$, $A \cdot B \cdot C \cdot D = \emptyset$. Classification is thus associated with the unification and intersection of extensions of concepts.

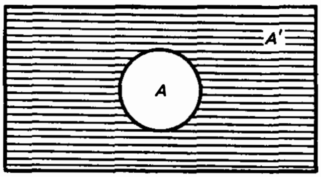


Fig. 34

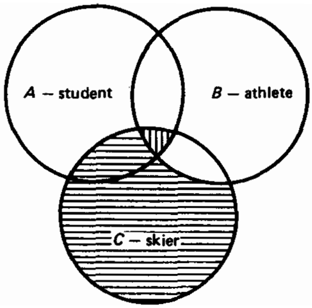


Fig. 35

The generalisation and limitation of concepts involves a complementation operation, since by complementing A to obtain 1 (the universal), we are making a transition from a species to a genus. We obtain in so doing a class A' which obeys the previously established laws of complementation ($A + A' = 1$ and $A \cdot A' = \emptyset$). For example, we may have the concept "town" (B) and generalise it to "place" (A). In so doing, we form the difference $A - B$, which denotes places that are not towns, i. e., the concept "not town" (hamlet, village, farm, settlement, etc.). This example shows that complementation is a particular instance of subtraction from a universal class. When limiting a concept, e. g. "boat" (C) to "submarine" (D), we carry out a transition from a genus to one of its species, making the subtraction

$C - D' = D$ or, in another notation $C - not\text{-}D = D$. In other words, we subtract from class C the complement to class D, i.e., we subtract the class "nonsubmarine boat" from the class "boat" to obtain "submarine". By adding (uniting) D and D', we obtain C.

We may use Euler's circles to express the relations between different concepts, e.g. "student" (A), "athlete" (B) and "skier" (C) (See Fig. 35).

In analysing the sections of these circles, we are carrying out various operations with classes. (Admittedly, we are doing this as yet without using the theory of unification, intersection and subtraction operations, since these come somewhat later in the section *Concept*.)

The vertically shaded section denotes the intersection of all three classes ($A \cdot B \cdot C$), that is, students who are both skiers, and athletes. The section of circle C with horizontal shading denotes the difference between classes C and B ($C - B$), i.e., skiers but not athletes. We may similarly analyse all possible variants (sections) obtained with the given location of the three circles A, B and C. The unification ("sum") of the three classes ($A + B + C$) includes all students, all athletes and all skiers.

Let us refer to another of the uses for operations with classes when dealing with concepts. The laws of idempotency ($A + A = A$ and $A \cdot A = A$) are used in the literary and scientific editing (shortening) of texts; on the basis of these laws we try to delete one instance of a concept used twice in a sentence by compressing our thoughts and the words used to express them.

Operations with classes (unification, intersection, subtraction and complementation) are thus used widely and variedly when dealing with concepts, including in the teaching process.

Exercises

I. Determine the intension, extension, subclasses of the extension and elements of the extension in the following concepts (inverted commas omitted): planets of the solar system; man who lived 205 years; chemical element; voltmeter; university faculty; Ohm's law; mainland; William Shakespeare; the trajectories of the planets of the solar system in 1982.

II. Give the logical characteristics of the following concepts: pupil's active attitude to life; heroic generation; South Pole; negligence; *Moscow News*, efficiency; quality; working masses; weightlessness; Konstantin Tsiolkovsky; impoliteness; 1980 Olympics; inorganic substance; lack of appropriate foresight.

III. Determine the relations between the following concepts:

1. Provision of assistance to a sick person, non-provision of assistance to a sick person.
2. Stone-built house, three-storey house; one-storey house, incomplete house.
3. Respect for one's elders, disrespect for one's elders.
4. Heroism, cowardice.
5. Teacher-training institute, biology faculty.
6. Mother, grandmother, granddaughter, sister.
7. Planetary satellite, natural satellite, Earth satellite, Jupiter, satellite of Jupiter, Moon.
8. Fire, lightning, natural disaster, natural phenomenon.
9. Fire, cause of fire, atomic bomb explosion, arson.

IV. Characterise (indicate the type, composition and correctness) of the following definitions:

1. Dentine is a special substance covering the teeth.
2. The outer ear is a helix.
3. Regeneration is the process of recovery of lost or damaged parts of the body.
4. A genre is a stable form of artistic production.
5. A writer's world outlook is his system of views on reality.
6. A fraction whose numerator is less than its denominator is called a proper fraction.
7. Archaisms are words which have gone out of use due to their replacement by new ones.
8. The hand is an organ and a product of labour.
9. Botany is the science devoted to the study of plants.
10. A bone is an organ with a complex structure.
11. The liver is a large organ weighing about 1.5 kilograms.
12. Phraseology is that part of the science dealing with the language which studies the semantic and

structural peculiarities of phraseological units, their types and the ways they function in speech.

13. A round cylinder can be obtained by rotating a rectangle around one of its sides, and for this reason a round cylinder is also called a cylinder of rotation.

14. The ending is the changeable part of a word by means of which a certain grammatical form is built which expresses the grammatical subordination of the given word to another word.

15. The path followed by blood from the left ventricle through the arteries, capillaries and veins of all organs to the right atrium is called the greater blood circulation circuit.

16. The term "impersonal" is applied to those sentences whose predicate does not admit of a subject.

17. Direct speech is exactly reproduced speech conveyed by the person from whom it originates.

18. A sphere may be obtained by rotating a semicircle (or a circle) around its diameter.

V. What forms of introducing concepts are used in the examples below?

1. "A duty to one's country is a sacred thing to man. It depends on us, fathers and mothers, teachers, to ensure that every young citizen of ours cherishes this sacred cause as an honest man cherishes his good name and the reputation of his family" (V. A. Sukhomlinsky, *On Education*, Leningrad, 1975, p. 219, in Russian).

2. The hypophysis is located in a depression in the main skull bone resembling a saddle.

3. The human heart has four chambers. In a state of relative calmness, the heart contracts rhythmically about 70 to 75 times a minute. The contraction of both auricles lasts about 0.1 seconds. The heart weighs about 300 grams.

4. Figuratively speaking, honey is a piece of sun on one's plate.

5. Blood serum is plasma with the fibrinogen removed.

6. Natural components are mountain rocks and their surface relief, water, air, flora, fauna and the soil.

7. Imagine an impervious equatorial forest. Huge trees stand like ancient fortresses, lianas, resembling thick cables, hang like bridges between the tops of

the trees at a dizzy height. Here grow huge bright mushrooms, and sharp-smelling flowers.

Small parasitic plants of rare beauty have perched on the branches of a huge tree; gradually they cover its trunk with their web of roots and strangle the tree until it dies.

VI. Characterise (give the type, composition and correctness) the following divisions and classifications. Point out any errors which may be present.

1. Volcanoes are divided into active, extint, dormant, central and cracked.

2. Triangles are divided into right-angled, acute, obtuse, equilateral and isosceles.

3. Cells may be spherical, disc-shaped, prismatic, cubic, spindle-shaped and polyhedral.

4. The skeleton of a bird's wing consists of one upper arm bone, two forearm bones – the ulna and radius – and the manus.

5. Seed plants are divided into gymnospermae and angiospermae.

6. Glands are divided into those of internal and of external secretion.

7. In the evolution of the organic world, we distinguish between two types of selection – natural and artificial.

8. Alkalis are divided into active and inactive.

9. In terms of designation transport is divided as follows:

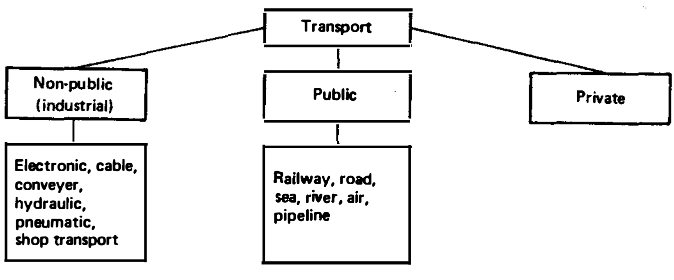


Fig. 36

10. Rays are divided into ultraviolet, visible and infrared.

6–570

11. In terms of their mechanical composition, soils are divided into loams, loamy soils, sands and sandy soils.

12. The main structural elements of a game are the intent, the subject of the game, or its content, moves, roles and rules.

13.

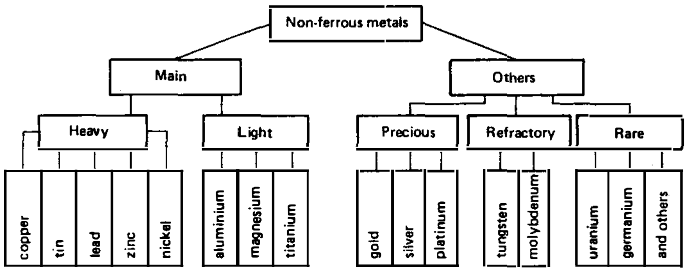


Fig. 37

14. There exist several types of thermal engines: steam engine, internal combustion engine, steam turbine, gas turbine, jet engine.

15. Toys are divided into figurative, technical, fancy dress, sporting toys, theatrical, didactical, building materials and home-made toys.

VII. 1. Generalise and limit the following concepts: river; geometrical figure; city in Asia; wheat; European; acid.

2. Are the following limitations correct: building – room; building – summerhouse; populated area – capital – centre of capital – centre of modern capital?

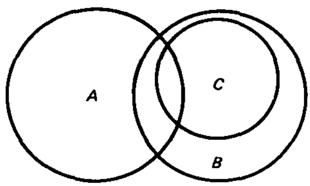


Fig. 38

VIII. 1. Carry out operations with classes *A*, *B* and *C* depicted in Fig. 38 (unification, intersection, subtraction).

2. Use Euler's circles to illustrate the relationship between the concepts: peasant, corn-grower, gardener. Carry out operations using these concepts (unification, intersection, subtraction); find a complement to each of these classes and indicate its universal.

Chapter III
JUDGEMENTS

§ 1. The General Nature of Judgement

Judgement is a form of thought in which something is affirmed or denied about the existence of objects, the links between an object and its properties or the relations between objects.

Examples of judgements are: "Icebreakers exist", "Soviet climbers ascended Everest", "London is bigger than Liverpool", "All engines are machines", "Some trees do not bear leaves". If a judgement asserts (or denies) the presence of some feature in an object, or refers to the existence of some object, or establishes a relationship between objects, the judgement is true if this accords with reality. The judgements "All grass snakes are reptiles", "10 is greater than 3", "Mermaids do not exist", "Some birds are not waterfowl" are true because they adequately (truthfully) reflect what happens in reality. Otherwise a judgement is false.

Traditional logic has two values, since in this logic a judgement has one of two values of truthfulness: it is either true or false. In three-valued logics, a judgement has one of three values, since it may be true, false or indeterminate. For example, the judgement "There is life on Mars" is neither true nor false at the present time; it is indeterminate. Many judgements about single future events are indeterminate. Aristotle wrote about this as long ago as the 4th century B.C., giving an example of this kind of judgement, "... a sea-fight must either take place on the morrow or not".[1]

[1] Aristotle, *On Interpretation,* I, IX, William Heinemann Ltd., London, 1980, p. 139.

In a simple judgement we find a subject, a predicate, a copula and a quantifier. In the judgements, "Some power stations are atomic power stations" and "All men are humans" the respective subjects are "power stations" and "men", the predicates are the concepts "atomic power station" and "humans", the quantifiers are "some" and "all" and the copula is expressed in both cases by the word "are". In the judgement "Icebreakers exist", the subject is the concept "icebreaker" and the predicate is the concept ("exists") referring to the existence of the subject.

The subject of a judgement is a concept referring to the object of the judgement. The predicate is a concept about the property of the object considered in the judgement. The subject is denoted by the letter *S* (from the Latin *subjectum*) and the predicate by the latter *P* (from the Latin *predicatum*). The copula may be expressed by a single word (is, are), a group of words or a simple word sequence ("The dog barks", "It hurts"). Sometimes we find a quantifier in front of the subject, such as "all", "none", "some", etc. The quantifier indicates whether the judgement refers to the entire extension of the concept used to express the subject, or just to part of it. The simple judgements referred to here are called *assertoric judgements.*

Judgement and sentence

Linguistically, concepts are expressed by one word or a group of words. Judgements are expressed by narrative sentences containing a piece of information. For example, "The storm is darkening the sky", "Many volcanoes are extinct", "No dolphin is a fish". According to the purpose of the statement, sentences are divided into narrative, hortative and interrogative.

Interrogative sentences do not contain any judgement, because they neither affirm nor disaffirm anything and are neither true nor false. Examples, "How do we use our free time?", "When will the table tennis competitions take place?". If a sentence expresses a rhetorical question, for example, "What Russian does not like fast travel?" (Gogol) or "Is there anything more monstrous than an ungrateful person?", or "Could anyone among you not love Pushkin's poetry?" or "Who doesn't want to be happy?", it contains a judgement, since it asserts, expresses certainty, that "Everyone loves Pushkin's poetry" or "Everyone wants to be happy", etc.[1]

Hortative sentences express the motivation of an interlocutor (reader and others) to carry out a certain act (the sentence may express a piece of advice, a request, an appeal, an order, etc.). A hortative sentence does not contain any judgement ("Wait for me", "Pour out the water"), although it does affirm something ("Protect the forest") or disaffirm it ("Don't pour out the water", "Don't go to the skating rink, but to school"). Sentences containing military commands, orders, appeals or slogans express judgements which are not assertoric but modal.[2] For example, "Workers of all countries, unite!". Judgements are also expressed by hortative sentences such as "Keep the peace!", "Do not smoke".

Impersonal sentences (e.g. "It is snowing", "It is dark"), nominative sentences (e.g. "Morning", "Autumn") and some types of narrative sentences (e.g. "*He* is an excellent cook", "The Far East is a long way *from us*") are judgements only when examined in a concrete context and specified, "Who is 'he'?", "Who is 'us'?". If this specification is absent, it is uncertain whether the sentence in question is true or false.

In some cases, the subject of a judgement (*S*) does not coincide with the grammatical subject and the predicate of a judgement (*P*) is not the same as its grammatical predicate. In the example, "Students are people engaged in study", they fully coincide. In the example, "*The Indian press gives great attention to the issue of oil*" (Here, straight underlining is given for the grammatical subject and predicate and a wavy line for the logical subject and predicate. As we see, they do not coincide. In the judgement, "*A big dog ran towards me*", they also do not coincide.

[1] For interrogative sentences and the role of the question in cognition, see Ch. VI.

[2] Modal judgements are examined in detail in § 6. They include modal statements expressed by the words: "it is possible", "it is necessary", "it is forbidden", "it is proven", etc.

§ 2. Simple Judgements

Judgements may be simple or complex, with the latter being made up of several simple ones. The judgement "Citizens of the USSR are guaranteed inviolability of the home" is a simple one, whilst the judgement "The black cloud came close, and we saw the first lightnings and heard thunder above our heads" and "The privacy of citizens, and of their correspondence, telephone conversations and telegraphic communications is protected by law" are complex judgements.

Types of simple judgements

1. *Judgements of properties (attributative)*. Judgements of this type affirm or deny the presence in an object of certain properties, states, types of activity. Examples, "Roses have a pleasant smell", "The singer performs an aria from the opera *Eugene Onegin*", "All terriers are dogs", "Three is not an even number". The formula of this type of judgement is: *S* is *P* or *S* is not *P*.

2. *Judgements of relations*. These judgements refer to relations between objects. For example, "Any proton is heavier than an electron", "The Elbrus is higher than Mont Blanc", "Fathers are older than their children", etc.

The formula expressing a judgement of relations is written as *aRb* or *R* (*a*, *b*), where *a* and *b* are the names of objects and R is the name of the relation. Judgements on relations may affirm or disaffirm something about two, three, four or even more objects. For example, "Moscow is located between Leningrad and Kiev".

3. *Judgements of existence (existential)*. These affirm or deny the existence of objects (material or ideal) in reality. For example, "There exists the *Lenin* atomic icebreaker", "There exist no phenomena without a cause".

4. *Categoric judgements and their types* (division according to quantity and quality). In traditional logic, attributive judgements are also called *categoric*. Depending on the nature of the copula ("is" or "is not") categoric judgements may be affirmative or negative. The judgements, "In many cities in various countries

demonstrations are held in defence of peace", "The defence of the socialist homeland is the sacred duty of every citizen of the USSR" are affirmative. The judgements "Some homes have no modern conveniences", "A crucian carp is not a predatory fish" are negative. In an affirmative judgement, the copula in the affirmative is used to reflect the presence of certain properties in an object (objects). The copula in the negative expresses the fact that a certain quality is not attributable to the object (objects) in question.

Some experts on logic believe that negative judgements contain no reflection of reality. In actual fact, the absence of a certain property is also a real property with an objective significance. In a true negative judgement, our thought separates what is separated in the objective world.

Depending on whether they refer to a whole class of objects, part of this class or one object, judgements may be *universal, particular* and *individual*. For example, "All chameleons are lizards" is a universal judgement; "Some flowers are roses" is a particular judgement; "Wolfgang Amadeus Mozart was a brilliant Austrian composer" is an individual judgement.

The structure of a universal judgement is: "All *S* are (are not) *P*".

Among universal judgements, we find *distinguishing* judgements, which include the quantifier "only". "Only a kind person can be a good doctor", "Pythagoras' theorem is only applicable to equilateral triangles".

Some universal judgements are also *exclusive*. For example, "All students in our group, with the exception of those who were sick, turned up for the *subbotnik*". Exclusive judgements also include those which express exceptions to linguistic rules, the rules of logic, mathematics and other disciplines.

Particular judgements have the structure: "Some *S* are (are not) *P*". They may be definite or indefinite. For example, "Some mushrooms are edible" is an indefinite particular judgement. We have not established whether all mushrooms display the property of edibility and have also failed to establish that some mushrooms do not display the property of edibility. If we had established that "Only some *S* display property *P*", this would be a definite judgement with the structure "Only some *S* are (are not) *P*". Examples: "Only some mushrooms are edible"; "Only some acute triangles are equilateral"; "Only some bodies are lighter than water". In definite particular judgements, we often encounter quantifiers: majority, minority, a lot, not all, many, almost all, several, etc.

Individual judgements have the structure: "This *S* is (is not) *P*". Examples of individual judgements are "Everest is the highest mountain in the world", "The Klyuchevsky volcano is active".

Combined classification of simple categoric judgements according to quantity and quality

Every judgement has its quantitative and qualitative dimension. For this reason, logic uses the combined classification of judgements according to quantity and quality as a basis for distinguishing between the following four types of judgement.

A is a universal affirmative judgement. Its structure is: "All *S* are *P*". For example, "All human beings are vertebrates".

I is a particular affirmative judgement. Its structure is expressed as: "Some *S* are *P*". For example, "Some fundamental particles have a positive charge". The conventional denotation for affirmative judgements is taken from the Latin *affirmo*. In practice we use two vowels: *A* – to denote a universal affirmative judgement and *I* to denote a particular affirmative judgement.

E is a universal negative judgement. Its structure is: "No *S* is *P*". For example, "No dolphin is a fish".

O is a particular negative judgement with the structure: "Some *S* are not *P*". For example, "Some workers are not builders". The conventional denotations for negative judgements are taken from the Latin *nego*.

Distribution of terms in categoric judgements

In judgements, the terms *S* and *P* may be distributed or non-distributed. A term is considered distributed if its extension is included completely in the extension of the other term or entirely excluded from it. A term will be non-distributed if its extension is partially included in the extension of the other term or partially excluded from it. Let us analyse the four types of judgements *A*, *I*, *E* and *O*.

Judgement A is universal and affirmative. Its structure is: "All *S* are *P*". Let us look at two instances.

First instance. In the judgement "All crucian carp are fish", the subject is the concept "crucian carp" and the predicate the concept "fish". The universal quantifier is "all". The subject is distributed, since we are concerned with all crucian carp, i. e., its extension is entirely included in the extension of the predicate. The predicate is not distributed, since the judgement refers to that part of the predicate which coincides with the extension of the subject.

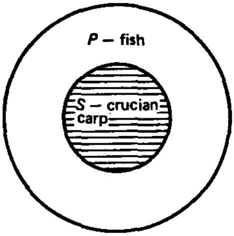


Fig. 39

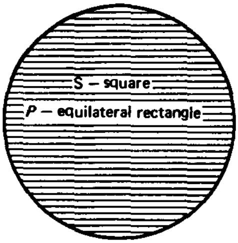


Fig. 40

The distribution of terms in judgements may be illustrated by using Euler's circles. Fig. 39 is a depiction of the correlation of *S* and *P* in judgement *A*. The shaded part of the circle in figures 39 to 44 is used to depict the distribution (or non-distribution) of terms.

If the extension of *P* is greater than the extension of *S*, then *P* is not distributed.

Second instance. In the judgement, "All squares are equilateral rectangles", the terms are as follows: *S* – "square", *P* – "equilateral rectangle", universal quantifier – "all". In this judgement, both *S* and *P* are distributed, since their extensions fully coincide (Fig. 40).

If *S* is equal in extent to *P*, then *P* is distributed. This occurs in definitions and distinguishing universal judgements.

Judgement I is particular and affirmative. Its structure is: "Some *S* are *P*". Let us examine two instances.

First instance. In the judgement, "Some students are stamp collectors," the terms are as follows: *S* – "students", *P* – "stamp collector", existential quantifier – "some". The correlation of *S* and *P* is depicted in Fig. 41. The subject is not distributed, since it refers only to a section of students, i. e., the extension of the subject is only partly included in the extension of the predicate. The predicate is not distributed either, since it too is only partly included in the extension of the subject (only some stamp collectors are students).

If the concepts *S* and *P* intersect, *P* is not distributed.

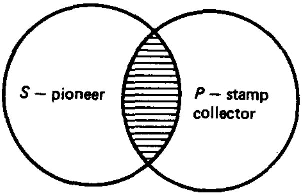


Fig. 41

Fig. 42

Second instance. In the judgement, "Some writers are dramatists", the terms are as follows: *S* – "writer", *P* – "dramatist", existential quantifier – "some". The subject is not distributed, since it refers only to a part of all writers, i. e., the extension of the subject is only partly included in the extension of the predicate. The predicate is distributed, since the extension of the predicate is fully included in the extension of the subject (Fig. 42). Thus *P* is distributed if the extension of *P* is less than the extension of *S*, as occurs in particular distinguishing judgements.

Judgement E is universal and negative. Its structure is: "No *S* is *P*". For example, "No lion is a herbivorous animal". The terms are as follows: S – "lion", P – "herbivorous animal", universal quantifier – "no". Here, the extension of the subject is completely excluded from the extension of the predicate, and vice versa. This means that both S and P are distributed (Fig. 43).

Judgement O is particular and negative. Its structure is: "Some S are not P". For example, "Some students are not athletes". Here, we have the following terms: S – "student", P – "athlete", quantifier of existence – "some". The subject is not distributed, since it only refers to part of students, but the predicate is distributed, referring to all athletes, none of whom is included among those students who form the subject (Fig. 44).

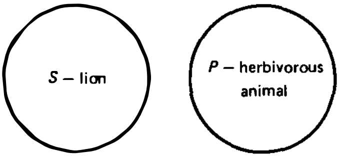


Fig. 43

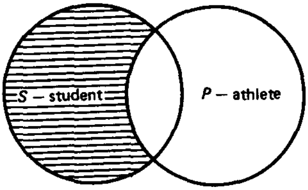


Fig. 44

To sum up, *S is distributed in universal judgements and not in particular ones; P is always distributed in negative judgements and distributed in affirmative ones when, in terms of its extension,* $P \leqslant S$.

The distribution of terms in categoric judgements may be expressed as follows, where the sign "+" is used to indicate that a term is distributed and the sign "–" that it is not distributed. The diagram gives summary information on simple judgements.

§ 3. Complex Judgements and Their Types

Complex judgements are formed from simple ones with the addition of logical connectives: conjunction, disjunction, implication, equivalence and negation. The tables depicting the truthfulness or falsity of these logical connectives are as follows:

Judgement

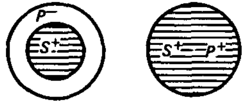
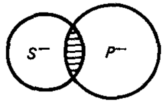
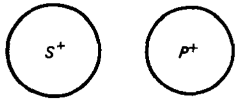
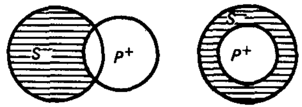
Type of judgement	Denotation	Formula of judgement		Distribution of terms in judgement		Relations between S and P
		in traditional logic	in mathematical logic (predicate calculus)	S	P	
Universal affirmative	A	All S are P (S a P)	$\forall x(S(x) \rightarrow P(x))$	$+$	$\mp$	
Particular affirmative	I	Some S are P (S i P)	$\exists x(S(x) \wedge P(x))$	$-$	$\mp$	
Universal negative	E	No S is P (S e P)	$\forall x(S(x) \rightarrow \overline{P(x)})$	$+$	$+$	
Particular negative	O	Some S are not P (S o P)	$\exists x(S(x) \wedge \overline{P(x)})$	$-$	$+$	

Fig. 45

a	b	$a \wedge b$	$a \vee b$	$a \dot{\vee} b$	$a \to b$	$a \equiv b$
T	T	T	T	F	T	T
T	F	F	T	T	F	F
F	T	F	T	T	T	F
F	F	F	F	F	T	T

$\bar{a}$	$\bar{a}$
T	F
F	T

The letters *a*, *b* and *c* are variables used to denote judgements; the letter "T" denotes a truth and the letter "F" a falsehood.

The table of truthfulness for conjunction ($a \wedge b$) may be explained in the following way. A short reference was given of a teacher, consisting of two simple judgements: "He is a good educationalist (*a*) and is doing a correspondence course (*b*)". The reference will be true only if both *a* and *b* are true. This is reflected in the first row. If *a* is false, or *b* is false, or both *a* and *b* are false, the entire conjunction will be false, i. e., the teacher will have been given a false reference.

The judgement, "Profitability may be raised by raising the productivity of labour (*a*) or cutting the cost of production (*b*)" is an example of inclusive disjunction. Disjunction is said to be inclusive if its elements are not mutually exclusive. Such statements are true if only one of the two judgements is true (first three rows in table) and false when both judgements are false.

The elements of exclusive disjunction ($a \dot{\vee} b$) are mutually exclusive. This may be illustrated with the aid of an example: "I will go to the South by train (*a*) or by air (*b*)". I cannot at one and the same time travel by train and by air. Exclusive disjunction is true when only one of the two simple judgements is true.

The table for implication ($a \to b$) may be illustrated with the following example: "If electric current is passed through a conductor (*a*), the conductor becomes hot (*b*)".[1]

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[1] Here, we are ignoring the difference between implication in propositional logic and the conjunction "if... then". One should also distinguish between conditional proposition expressed by the indi-

Implication is always true, except in one case, namely when the first judgement is true and the second one is false. Indeed, it would be impossible to pass an electric current through a conductor, i. e., for judgement (*a*) to be true, without the conductor becoming hot, i. e., for judgement (*b*) to be false.

Equivalence ($a \equiv b$) is characterised as follows in the table: $a \equiv b$ is true only if *a* and *b* are both true or they are both false.

The negation of judgement *a* (i. e., $\bar{a}$) is characterised by the fact that if *a* is true, $\bar{a}$ is false and, if *a* is false, then $\bar{a}$ is true.

If the formula included three variables, comprising all combinations of truthfulness and falsehood for these variables in a table, it would consist of $2^3 = 8$ rows; for 4 variables, it would have $2^4 = 16$ rows, for five variables, 2^5 rows and for *n* variables, 2^n rows (see following two tables).

The algorithm of the distribution of *T* and *F* values (for example, for four variables *a*, *b*, *c* and *d*) is as follows:

a	*b*	*c*	*d*
T	T	T	T
T	T	T	F
T	T	F	T
T	T	F	F
T	F	T	T
T	F	T	F
T	F	F	T
T	F	F	F
F	T	T	T
F	T	T	F
F	T	F	T
F	T	F	F
F	F	T	T
F	F	T	F
F	F	F	T
F	F	F	F

We have $2^4 = 16$ rows.

cative and subjunctive moods. The latter are called unreal propositions. For example, "If there were no oxygen on Earth, it would be impossible for any life to exist on it".

In the columns for a, we first write "T" 8 times and then "F" 8 times.

In the column for b, we write "T" 4 times, "F" 4 times and then repeat the operation, etc.

An identically true formula is one which always takes the value "true" for any combinations of variables. An identically false formula is accordingly one which always takes the value "false". A fulfillable formula may be taken as having the value "true" or the value "false".

Let us provide demonstration of the invariable truth of the formula $((a \to (b \wedge c)) \wedge (\bar{b} \vee \bar{c})) \to \bar{a}$.

a	b	c	$\bar{a}$	$\bar{b}$	$\bar{c}$	$b \wedge c$	$a \to (b \wedge c)$	$(\bar{b} \vee \bar{c})$	$(a \to (b \wedge c)) \wedge (\bar{b} \vee \bar{c})$	$((a \to (b \wedge c)) \wedge (\bar{b} \vee \bar{c}) \to \bar{a}$
T	T	T	F	F	F	T	T	F	F	T
T	T	F	F	F	T	F	F	T	F	T
T	F	T	F	T	F	F	F	T	F	T
T	F	F	F	T	T	F	F	T	F	T
F	T	T	T	F	F	T	T	F	F	T
F	T	F	T	F	T	F	T	T	T	T
F	F	T	T	T	F	F	T	T	T	T
F	F	F	T	T	T	F	T	T	T	T

Since we only find the value "true" in the last column, the formula is identically true, or a law of logic (such expressions are called tautologies).

Conclusions

Conjunction ($a \wedge b$) is true when both simple judgements are true. Exclusive disjunction ($a \veebar b$) is true only when one simple judgement is true. Inclusive disjunction ($a \vee b$) is true when at least one simple judgement is true. Implication ($a \to b$) is true in all cases but one: when a is true and b is false. Equivalence ($a \equiv b$) is true when both judgements are true or both are false. The negation ($\bar{a}$) of a truth yields a falsehood, and vice versa.

Ways of negating judgements

Two judgements are called *negating* or *contradictory* to each other when one of them is true and the other is definitely false (i. e., they cannot be true at the same time or false at the same time). The following pairs of judgements are negating.

a	$\bar{a}$
T	F
F	T

1. $A - O$. "All S are P" and "Some S are not P"
2. $E - I$. "No S is P", "Some S are P".
3. "This S is P", "This S is not P".

The operation of negation, when it involves forming a new judgement from one already given, should be distinguished from negation as a part of negative judgements. There are two types of negation: internal and external. Internal negation occurs when the predicate does not accord with the subject (the copula is expressed as "is not"). For example, "Some people *do not have* higher education". External negation denotes the negation of the entire judgement. For example, "It *is not true* that the River Neva flows through Moscow".

Negation of complex judgements

In order to obtain a negation of a complex judgement composed just of conjunction and disjunction operations, the signs for the operations in question have to be replaced with their opposites (i. e., conjunction by disjunction, and vice versa). A negation sign must be placed above the letters expressing elementary propositions or, if it is already there, it must be removed. We have: 1) $\overline{a \vee b} \equiv \bar{a} \wedge \bar{b}$. 2) $\overline{\bar{a} \vee \bar{b}} \equiv a \wedge b$. 3) $\overline{a \wedge b} \equiv$ $\equiv \bar{a} \vee \bar{b}$. 4) $\overline{\bar{a} \wedge \bar{b}} \equiv a \vee b$. These four formulas are called De Morgan's laws. By applying them, we obtain: $\overline{(a \vee \bar{b}) \wedge (c \vee e)} \equiv (\bar{a} \wedge b) \vee (\bar{c} \wedge \bar{e})$.

If a complex judgement contains an implication, it must be replaced by an identical formula without an implication (with a disjunction), i. e., $(a \rightarrow b) \equiv (\bar{a} \vee b)$; and then the opposite judgement can be found by the general method. For example, "If I have some free time (a), I shall read a book (b) or watch television (c)". The formula of this complex judgement is $a \rightarrow (b \vee c)$. The contradictory judgement will be $\overline{a \rightarrow (b \vee c)} \equiv$

$\overline{\bar{a} \vee (b \vee c)} \equiv a \wedge (\bar{b} \wedge \bar{c})$. It reads as follows, "I shall have some free time, but I will not read a book and will not watch television".

§ 4. The Expression of Logical Connectives (Logical Constants) in Natural Language

In the process of thought, we do not merely rely on simple, but also on complex, judgements, the latter being built from the former by means of logical connectives (or operations), such as conjunction, disjunction, implication, equivalence and negation. These operations are also known as logical constants. Let us analyse in what way the logical connectives listed are expressed in natural language.

Conjunction (denoted by "$\wedge$") is expressed by the conjunctions "and", "but", "although", "which", "however", "not only... but also", etc. In propositional logic, the sign "$\wedge$" connects simple judgements to form complex ones. In natural language, the conjunction "and" as well as others, which accord with conjunction, may link nouns, verbs, adverbs, adjectives and other parts of speech. For example, "In granddad's basket there were mushrooms and berries" ($a \wedge b$), "There is an interesting and beautifully designed book lying on the table". The latter judgement may not be split into two simple combined conjunctions: "There is an interesting book lying on the table" and "There is a beautifully designed book lying on the table", since this creates the impression that there are two books rather than one lying on the table.

The law of the commutativity of conjunctions $(a \wedge b) \equiv (b \wedge a)$ applies in propositional logic. This law does not exist in natural language due to the effect of the time factor. For this reason, the following two judgements are not equivalent: 1) "The steam engine was coupled up and the train set off", 2) "The train set off and the steam engine was coupled up".

In natural language, conjunction may not only be expressed in words, but also in punctuation marks, such as commas, semicolons and dashes. For example, "It flashed with lightning, roared with thunder, began to rain".

In propositional logic, a conjunction is true when all the judgements it links are true. Account is only taken of the truth value of the simple judgements; a link in terms of sense between the simple judgements may be absent. For example, in the calculus of propositions, the following statement is a true one, "A cow is a mammal, and $5 \times 6 = 30$", whilst in natural language we do not construct such complex sentences, since they have no sense.

Stephen Cole Kleene deals with the expression of conjunction in his book *Mathematical Logic*, and in the section on the analysis of arguments gives a (inexhaustive) list of expressions from natural language which may be replaced by the symbol "$\wedge$" or "&". In natural language, the formula $A \wedge B$ may be expressed as follows:

"Not only *A* but *B*.
B though also *A*.
B despite *A*.
Both *A* and *B*.
A and *B*.
A while *B*."[1]

We leave the reader to think up examples of all these structures.

In natural language, disjunction (denoted by $a \vee b$ and $a \dot{\vee} b$) is expressed by words like "or" and "either ... or". For example, "Tonight I'll go to the cinema or the library; "This animal belongs either to the vertebrates or to the invertebrates"; "The lecture will either be on the works of Tolstoy or those of Dostoyevsky".

The law of commutativity applies to both types of disjunction: $(a \vee b) \equiv (b \vee a)$ and $(a \dot{\vee} b) \equiv (b \dot{\vee} a)$. This equivalence is maintained in natural language. For example, the judgement "I'll buy butter or bread" is equivalent to the judgement "I'll buy bread or butter".

Kleene demonstrates what a wide range of means can be found in natural language to express implication $(A \supset B)$ and equivalence $(A \sim B)$.[2] (The letters *A* and *B* are used to denote the propositions' variables.)

Let us give logical patterns and corresponding examples in order to illustrate the variety of ways to express

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[1] Stephen Cole Kleene, *Mathematical Logic*, John Wiley & Sons, New York, 1967, p. 63.

[2] *Ibid.*

implication, $A \to B$, where *A* is the antecedent and *B* is the consequent.

1. *If A, then B.*

If the suppliers deliver the parts on time, *then* the factory will fulfil its production plan.

2. *As soon as* A, *then* B.

As soon as the force on a tensed spring is released, *then* it returns to its original shape.

3. *When* A, B *occurs.*

When bad weather sets in, there *occurs* a growth in the number of cardiovascular illnesses.

4. *A* is *sufficient* for *B*.

It is *sufficient* to heat gases *for* them to expand.

5. *For A, B is necessary.*

For the preservation of peace on Earth, it is *necessary* to unite the efforts of all states in the struggle for peace.

6. *A only if B.*

The students did not turn up for the subbotnik *only if* they were ill.

7. *B if A.*

I'll let you take a walk *if* you do your homework.

We shall now give the logical patterns and relevant examples of various ways to express *equivalence.*

1. *A if and only if B.*

Ivanov will complete his experiments on time *if and only if* his workmates lend a hand.

2. *If A, then B, and vice versa.*

If a student passes all exams and his practical assignment with excellent marks, *then* he is awarded a degree with a distinction, *and vice versa.*

3. *A if B and B if A.*

A polygon is inscribed in a circle *if* its apexes lie on the circle, *and* the apexes of a polygon lie on a circle *if* the said polygon is inscribed in this circle.

4. *B is necessary* and *sufficient for A.*

It is *necessary* and *sufficient* for the sum of figures in a number to be divisible by 3 with no remainder *for* the number itself to be exactly divisible by 3.

5. *A is equivalent to B.*

To say that the area of a regular polygon is equal to its semi-perimeter multiplied by its apothem is *equivalent to* saying that the area of a regular polygon is equal to a perimeter multiplied by its semi-apothem.

6. *A if* and *only if B.*

The firm will be prepared to purchase the goods *if and only if* their price is reduced by 15 per cent.

In computer modelling of texts in natural languages, including negation, it is possible to write down some expressions in the algebra of logic (*A*, *B*, *C* and *D* are propositions, "+" signifies inclusive disjunction, "·" signifies conjunction and "–" denotes negation.

Verbal definition	*Logical statement*
Not not *A*.	$\bar{\bar{A}}$
Not *A* but *B*.	$\bar{A} \cdot B.$
Not only *A*, but also *B*.	$A \cdot B.$
A but not *B*.	$A \cdot \bar{B}.$
A, but not *B*, *C*, but not *D*.	$A \cdot \bar{B} \cdot C \cdot \bar{D}.$
Not what *A* represents, but *B*.	$\bar{A} \cdot B.$
Not for *A* but for *B*.	$\bar{A} \cdot B.$
A and not *B*.	$A \cdot B.$
Not *A*, not *B*, but *C*.	$\bar{A} \cdot \bar{B} \cdot C, \ \overline{A + B} \cdot C.$
Not *A*, not *B*, not *C*, but *D*.	$\bar{A} \cdot \bar{B} \cdot \bar{C} \cdot D, \ \overline{A + B + C} \cdot D.$
A, but neither *B* nor *C*.	$A \cdot \bar{B} \cdot \bar{C}, \ A \cdot \overline{B + C}.$
Neither *A* nor *B*.	$\overline{A \cdot B}, \ \overline{A + B}.$
Neither *A* nor *B*, but *C*.	$\bar{A} \cdot \bar{B} \cdot C, \ \overline{A + B} \cdot C.$
A or *B*, but not *C*.	$(A + B) \cdot \bar{C}.$

The above patterns and corresponding examples with their concrete and varied content, make it clear how wide is the range of means available in natural language (English) for expressing implication, equivalence and other logical connectives. This can also be said of other natural languages.[1]

Implication $(a \rightarrow b)$ does not entirely correspond to the sense of the expression "if ... then", since it may lack a meaningful link between judgements *a* and *b*. In propositional logic the law is $(a \rightarrow b) \equiv (\bar{a} \vee \bar{b})$, but in natural language the situation is somewhat different. Sometimes the expression "if... then" contains not

[1] We recommend you to independently examine the ways in which logical links are expressed in any other natural language or in the works of any writer.

implication but conjunction. "If it was dull yesterday, the sun is shining brightly today". This complex judgement is expressed by the formula $A \wedge B$.

Apart from connectives to express universal and particular judgements, logic employs universal and existential quantifiers. A formula employing the universal quantifier $\forall x P(x)$ usually reads like this: "All x (from some field of objects) have the property P", whilst a formula with the existential quantifier $\exists x P(x)$ looks like this: "There are x (in the given field) which have the property P". For example, $\exists x\ (x > 100)$ is read as "There are some x which are greater than 100", where x refers to numbers. In natural language, the universal quantifier is expressed by the words "all", "any", "every", "none", etc. The existential quantifier is expressed by: "some", "there exist", "majority", "minority", "only some", "sometimes", "that which", "not all", "many", "a lot", "not many", "much", "almost all", etc.

Stephen Kleene states that by translating expressions from natural language by means of the propositional connectives in the table we lose some shades of meaning but gain greater accuracy.

Unreal is the name given to conditional propositions expressed in the subjunctive mood. For example, "If there were no oxygen on Earth, it would be impossible to survive on it", "If the driver hadn't broken the rules, the accident wouldn't have happened."

The subjunctive indicates that the antecedent and consequent in the said propositions are false, i. e., they do not reflect the actual state of affairs. However, just like any other propositions, unreal propositions may be true when viewed in their entirety. They are true if the connective between the antecedent and the consequent is such that the truthfulness of the antecedent implies the truth of the consequent. For example, the statement "If it were night, it would be dark outside" is true, but the statement "If it were night, it would be light outside" is false (provided it does not apply to the Far North, where there is the midnight sun in summer). Since the antecedent and the consequent in an unreal proposition are both false, it is difficult to establish its truth.

An unreal proposition has the structure: "If A were the case, then B would be the case". In evaluating events, intentions, motives, political plans, etc., historians often make use of unreal propositions to say what could have happened if things had taken a different course to that they actually did. It is accepted practice to denote unreal propositions whose indicative forms of the antecedent and consequent are represented by A and B respectively as $A \mapsto B$.

The following true statement is an example of an unreal complex proposition: "The consequences of the calamity could have been worse if not for the courage people showed and the way they stuck together, the precise organisation of the rescue operations and the consistent fulfilment of their duties by all concerned". In order to write down the formula of this complex unreal proposition, we must first put it into a clear logical form. It is as follows: "If there had been a lack of courage on the part of people and they had not stuck together, if the rescue operations had not been precisely organised and everyone had not consistently fulfilled their duties, the consequences of the calamity could have been worse". The formula of this unreal statement is: $(A \wedge B \wedge \wedge C \wedge D) \mapsto E$. Here, A denotes the statement, "There was a lack of courage on the part of people", B "people did not stick together", C "the rescue operations were not precisely organised" and D "everyone did not consistently fulfil their duties". All four statements are linked by a conjunction sign. The sign '$\mapsto$' denotes implication in an unreal proposition corresponding to the expression, "if it were..., then it would be". The letter E denotes the statement "The consequences of the calamity were worse". It should be pointed out that the sign "$\mapsto$" is not found in classical propositional logic.

Unreal propositions are encountered quite frequently in both scientific literature and fiction.

Mathematical and other forms of reasoning employ the concepts "necessary condition" and "sufficient condition". A condition is called *necessary* if it derives from a conclusion (effect). A condition is called sufficient if a conclusion or effect follow from it. In the implication $a \to b$, variable a is the cause and variable b the effect (conclusion). Below we present some exercises in which the dotted lines are to be replaced by words: either "necessary" or "sufficient", or "necessary and sufficient".

1. In order for the sum of two whole numbers to be an even number it is ... for each component to be even.

2. In order for a number to be divisible by 15 it is ... for it to be divisible by 5.

3. In order for the product $(x - 3)(x + 2)(x - 5)$ to be equal to 0, it is ... for $x = 3$.

4. In order for a quadrangle to be a rectangle it is ... for all its angles to be equal.

§ 5. Relations Between Judgements in Terms of Truth Values

Like concepts, judgements may be comparable (have a common subject and predicate) or incomparable. *Comparable* judgements are divided into compatible and incompatible. Compatible judgements express one and the same thought either in full or in part. *The relations of compatibility are equivalence, logical subordination, partial coincidence (subopposition).* Compatible equivalent judgements express one and the same thought in different forms ("Yuri Gagarin was the world's first cosmonaut", "Yuri Gagarin was the first man to fly in space"). The subject here is one and the same, whilst the predicates merely differ in form, having the same meaning. In the two equivalent judgements, "Mikhail Sholokhov was a Nobel Prize winner", "The author of *Quiet Flows the Don* was a Nobel Prize winner", the predicates are identical and, whilst the subjects differ in terms of form, they too are identical concepts.

Whether one is composing a piece of writing, learning some material by heart, giving an oral interpretation of a text or translating from one language to another, one has to be able to formulate one's thoughts in a concise and correct form.

Compatible judgements which are connected by *logical subordination* have a common predicate; concepts expressing the subjects of two such judgements are also in a relation of logical subordination. It is accepted practice to express the relations between the truth values of the two judgements in the form of the "logical square" (Fig. 46).

Let us take the judgements "all elephants are domesticated". This judgement may be denoted by *A* as general and affirmative (subordinating). Judgement *I*: "Some elephants are domesticated" is subordinate.

Judgement *I* being subordinate to judgement *A*, and *O* subordinate to *E*, the truth of the universal judgement also entails the truth of the particular, subordinate judgement. But if the universal judgement is false, it is not clear whether the particular judgement will be true or false. The truth of a particular judgement leaves the truth value of the corresponding universal judgement

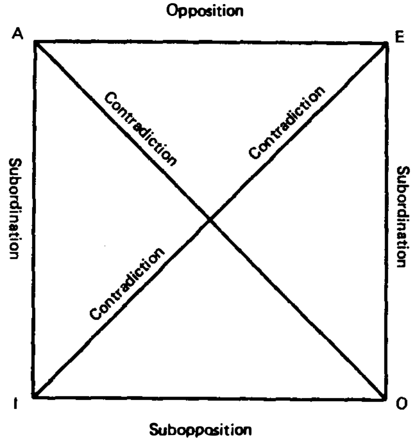

Fig. 46

indeterminate (if this rule is violated, it may lead to a logical error, "premature generalisation"). The falsehood of a particular judgement implies the falsehood of the universal judgement to which it is subordinate. If the judgement "No trapezium is a spherical body" is true, then the judgement "Some trapeziums are not spherical bodies" will also be true. Inference of a logically subordinate particular judgement from a universal one will always provide a true conclusion.

A relation of *partial coincidence* (*subopposition*) is said to obtain between two compatible judgements *I* and *O* which have the same subjects and the same predicates but differ in terms of quality. For example, *I* "Some witnesses give truthful evidence" and *O* "Some witnesses do not give truthful evidence". Both of these judgements may be true at the same time, but it is impossible for them both to be false. If one of them is false, the other must necessarily be true. But if one of them is true, the other is uncertain (it may be either true or false). For example, if judgement *I*, "Some books in this library are antiquarian", is true, judgement *O*, "Some books in this library are not antiquarian", will be indeterminate, i.e., it may be true or false.

Relations of incompatibility: opposition, contradiction. In terms of the logical square, judgements *A* and *E* are in *opposition* to each other. The two judgements: *A* – "All people work conscientiously" and *E* – "No people work conscientiously" may both be false. But *A* and *E* may not both be true. If one opposing judgement is true, the other will necessarily be false.

Thus, the truth of one opposing judgement entails the falsehood of the other, but the falsehood of one of them leaves the truth of the other uncertain.

Judgements *A* and *O*, and also *E* and *I*, are said to be in *contradiction* to each other. Two contradictory judgements cannot both be either true or false at the same time. If judgement *I* – "Some pilots are cosmonauts" is true at the present time, the judgement, "No pilots are cosmonauts" will definitely be false.

The laws expressing relations between judgements in terms of their truth values are of great cognitive significance, since they help avoid errors in direct inferences made from one premise (one judgement).

§ 6. The Division of Judgements by Modality

In terms of logic, we have so far examined simple judgements (also called *assertoric*) and complex judgements made up of simple ones. They are used to affirm or deny the presence of certain connections between an object and its properties, or to express a relation between two or a larger number of objects. For example, "In a right-angled triangle, the sum of the squares of the catheti is equal to the square of the hypotenuse, i.e., $a^2 + b^2 = c^2$"; "The volume of a cone is equal to 1/3 the area of the base, multiplied by the height"; "The apple is sweet and red"; "I will not complete this work on time"; "If the weather is bad, we shall not go on the ship", etc. The general form of such simple statements (judgements) is: "*S* is (is not) *P*". Simple judgements are combined to form complex ones, for example: "If *S* is (is not) *P*, then S_1 is (is not) P_1".

In these assertoric judgements, we do not establish the nature of the link between the subject and the predicate. The nature of the link between the subject and the predicate, or between individual simple judgements in a complex judgement, is revealed in modal judgements. It is possible, for example, to form the following modal judgements from those given above: "*It has been proven* that in a right-angled triangle the sum of the squares of the catheti is equal to the square of the hypotenuse", "*It is good* that an apple is sweet and red", "*It is possible* that I will not complete this work on time", "*It is probable* that if the weather is bad we shall not go on the ship". We can see that modal judgements do not simply affirm or deny some link, but give an assessment of this link from some viewpoint.

We may simply say of object *A* that it has property *B* (this will be an assertoric judgement). But we can go beyond that and specify whether this connection between *A* and *B* is necessary or, on the other hand, coincidental; whether it is a good or a bad thing that *A* is *B*; whether it is proven or not that *A* is *B*, i.e., whether it is merely supposed, etc. By introducing such clarifications, we obtain various types of modal judgements. Let us give some other examples of modal judgements: "*It is possible* that there is life on Mars", "*It is proven* that a limited nuclear war is impossible under present conditions", "*It is necessary* to have a social revolution in order to accomplish the transition from one socioeconomic formation to another", "*It is forbidden* for the traffic to pass when the traffic lights are at red", etc. In a modal judgement, we attach a modal operator (modal concept) to an assertoric judgement: it is possible, it is proven, it is necessary, it is forbidden, it is compulsory, it is bad, etc.

The structure of simple modal judgements is as follows:

$$M(S \text{ is } P) \text{ or } M(S \text{ is not } P),$$

where M is used to denote the modal operator (modal concept).

However, as already pointed out, complex judgements may also be modal. If a and b are simple assertoric judgements, then from the complex assertoric judgements: $a \wedge b$, $a \vee b$, $a \dot{\vee} b$, $a \rightarrow b$ and $a \equiv b$, we may obtain the corresponding complex modal judgements:

$$M(a \wedge b);\ M(a \vee b);\ M(a \dot{\vee} b);\ M(a \rightarrow b) \text{ and } M(a \equiv b).$$

In each of these five types of complex modal judgements, the modal operator M may be replaced by one of its varieties. For example, from the complex assertoric judgement: "If fertilizer is applied to the soil, the harvest will increase", we may obtain the following modal judgements: "*It has been proven* that, if fertilizer is applied to the soil, the harvest will increase", "*It is good* that, if fertilizer is applied to the soil, the harvest will increase".

A *simple modal judgement* is a simple judgement which expresses a link between the subject and the predicate by means of a modal operator (modal concept).

A *complex modal judgement* is a complex judgement which expresses a link between its component simple judgements by means of a modal operator (modal concept).

Modal statements are the subject of study by modal logic, which consists of the following sections (or branches): the logic of norms, the logic of time and others.

As things stand at present, modal logic has studied many types of modalities, and those which have been investigated with relative thoroughness are systematised in the following table drawn up by the Soviet scholar A. A. Ivin.

There are three fundamental modal concepts in each group of modalities. The second of them is called the weak characteristic, and the first and third are called the strong positive and strong negative characteristics respectively. Sometimes a fourth modal concept is in-

Logical modality	Ontological modality	Epistemic modality	
		knowledge	conviction
Logically necessary	Ontologically necessary	Provable (verifiable)	Believes (is convinced)
Logically accidental	Ontologically accidental	Insoluble (unverifiable)	Doubts
Logically impossible	Ontologically impossible	Disprovable (falsifiable)	Rejects
Logically possible	Ontologically possible		Considers possible

Deontic modality	Axiological modality		Temporal modality	
	absolute	comparative	absolute	comparative
Compulsory	Good	Better	Always	Earlier
Normatively indifferent	Axiologically indifferent	Equivalent	Only sometimes	Simultaneously
Forbidden	Bad	Worse	Never	Later
Allowed				

troduced in addition to the three fundamental ones and may be used instead of them.

Logical and ontological modalities are varieties of the same type, namely alethic modalities.[1] They also include modal operators, or categories of modality: necessity and accident, possibility and impossibility. In everyday language, the words "necessary", "possible" and "accidental" are used in many different senses.

[1] The term "epistemic modality" comes from the Greek word "episteme", which in ancient philosophy meant the highest type of undoubted, certain knowledge. The term "deontic" is also borrowed from the Greak and denotes compulsion. The term "alethic", likewise of Greek origin, is used to denote a necessity.

The philosophy of dialectical materialism studies the categories "necessity", "accident" and "possibility" from the substantive angle. Formal logic studies the formal relations existing between them. From the viewpoint of Marxist philosophy, the universality of modal categories stems from their applicability to any area of reality.

Alethic modalities are denoted as follows: "$\Box A$" – "necessarily A"; "∇A" – "accidentally A"; "$\Diamond A$" – "possibly A"; "$\sim \Diamond A$" – "A is impossible ". The sign "$\sim$" denotes negation. They are sometimes denoted as follows: "Lp" – "necessarily p", "Mp" – "possibly p".

Alethic modalities (logical and ontological) are often essentially interpreted as follows: logical laws are considered necessary, as are laws revealed by the various sciences (natural, mathematical, social and technical), and all consequences arising from these laws. Judgements which contradict these laws, denying them or their consequences, are considered impossible. Judgements which are not laws or their consequences, but do not contradict either the laws themselves or their consequences, are considered accidental. Propositions which do not contradict laws or their consequences are considered possible.

Links between alethic modalities

In certain systems, one alethic modality can be used to define another ("Df" denotes "equal by definition", "$\wedge$" conjunction, "$\vee$" disjunction, "$\sim$" negation, "$\leftrightarrow$" equivalence and "$\rightarrow$" implication). For example, $\Box A \underset{Df}{=} \sim \Diamond \sim A$.

Some important links characterising alethic modalities are expressed in the following formulas.

(1) $\Box A \rightarrow A$ ("If it is necessary that A, then A");

(2) $A \rightarrow \Diamond A$ ("If A, then it is possible that A").

(3) $\Box A \leftrightarrow \sim \Diamond \sim A$ (It is necessary that A when and only when it is impossible that *not-A*");

(4) $\Diamond A \leftrightarrow \sim \Box \sim A$ ("It is possible that A when and only when it is not necessary that *not-A*").

Table 1

The basis of classification – the form (structure of a judgement)

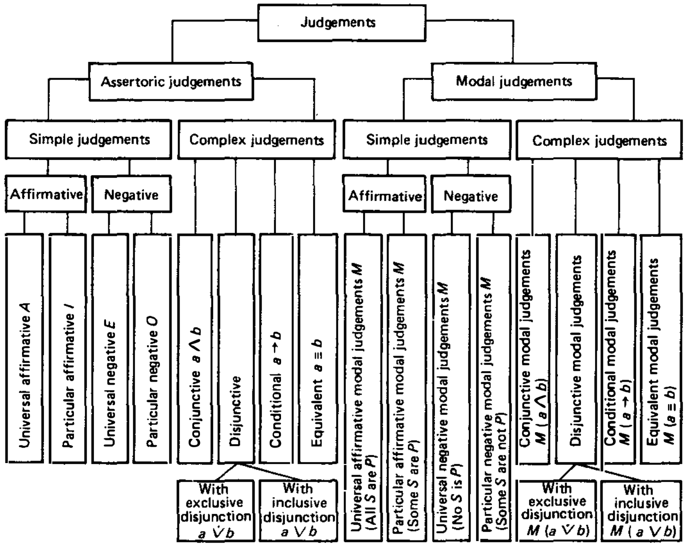

Fig. 47

Table 2

Basis of classification – content of a judgement

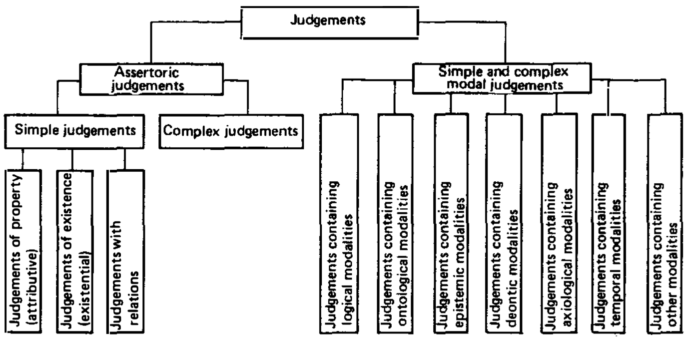

Fig. 48

Description of tables classifying judgements

Figure 47 is constructed in such a way as to classify judgements on the basis of their form or structure. The two main groups of judgements – assertoric and modal – are differentiated on the basis that the former does not establish the nature of the link between the subject and the predicate, or between the simple judgements that make up a complex one, whilst the latter establish the nature of the link between the subject and predicate in a simple modal judgement and that between the individual simple modal judgements which comprise a complex one.

There is also an analogy apparent in the structure of assertoric and modal judgements: first, both types of judgement may be simple or complex; second, simple judgements may be affirmative or negative; each of these in turn may be a universal or particular judgement, which gives us four types of simple assertoric judgements and four types of simple modal judgements; third, complex assertoric and complex modal judgements have a similar structure; each group includes conjunctive, disjunctive, conditional and equivalent judgements. Among assertoric judgements, no distinction is made between the distinguishing and the exclusive ones, since they do not quite fit into this table, being varieties of universal or particular judgements.

Figure 48 classifies judgements according to their content. Simple assertoric judgements are divided into three types: judgements referring to properties, those expressing existence and a third type dealing with relations. No distinctions are made between complex judgements, since they are the same as in *Figure 47* (i. e., complex conjunctive judgements, disjunctive, etc.).

Simple and complex modal judgements may be divided up on the basis of the type of modality contained in the judgement: judgements containing epistemic modalities, deontic modalities, etc. Judgements containing other modalities are given separately, since not all modalities have yet been studied in detail, and they cannot all be listed. The rules for dividing concepts (and hence the rules of classification) allow the introduction of other elements of division as a separate group when the number of elements is relatively large or when not all species of the generic concept being divided have been thoroughly studied or are well known.

Exercises

1. Are the following formulas laws of logic?

   1) $\overline{a \to b} \equiv a \wedge \bar{b}$;
   2) $\bar{a} \vee \bar{b} \equiv a \wedge b$;
   3) $\overline{a \wedge b} \equiv \bar{a} \vee \bar{b}$;
   4) $((a \to \overline{(b \wedge c)}) \wedge \overline{(\bar{b} \vee \bar{c})}) \to \bar{a}$.
   5) $((a \to \bar{b}) \wedge (\bar{c} \to d) \wedge (b \vee \bar{d})) \to (\bar{a} \vee c)$.

2. Define the type of judgement, its terms and their distributivity in the following judgements:
   a) People are sometimes late for work.
   b) All extended sentences have secondary parts.
   c) Some people are illiterate.
   d) When heated to 80°C, water does not boil.
   e) Social being determines social consciousness.
   f) No dolphins are fish.
   g) No oceans have fresh water.
   h) There is no excuse for impoliteness.
   i) No medicinal plants are inedible.
   j) Wednesday is the third day of the week.
   k) Some people do not study logic.

3. Determine the type and logical form of the following complex statements. Write down their structure as a formula.

   a) The hills are covered in foliage and the snow is melting from the volcanoes in June when trees and shrubs are planted in the squares and streets of Petropavlovsk-Kamchatsky and the flower beds are laid out.

   b) Their arrival is neither necessary nor desirable.

   c) If this figure is a square, its diagonals are equal, perpendicular and intersect in the middle.

   d) If you wish to have an indestructible memorial, then put your heart into a good book.

4. Give the negation of the following complex judgements, first writing down their structure as a formula.

   a) If I am given a holiday in summer, I shall either go to the seaside or on a package tour to the Carpathian Mountains.

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b) If a student belongs to a students' scientific society, he takes part in academic work and gives lectures on his chosen theme.

c) It is not true that this writer is a dramatist or a poet.

5. Determine the logical relations between judgements *A*, *E*, *I* and *O* using the "logical square".

a) All people are literate. Some people are literate.
Nobody is literate.
Some people are not literate.

Do the same for b, c and d. But first you will have to specify which three judgements are missing in each case.

b) No predicates are secondary parts of a sentence.

c) Some hockey players are Olympic champions.

d) Some subjects are expressed by a pronoun in the nominative case.

6. Use the "logical square" to determine the relations between the following simple judgements.

a) It is not true that all children are obedient – Some children are obedient.

b) All books are manuscripts – No books are manuscripts.

7. Determine the type of modality in the following judgements.

a) With the appearance of genetic engineering, we can expect considerable success in improving the quality and composition of microbiological production.

b) It is likely that milk was one of the first items of agricultural produce.

c) Bread baking emerged at the dawn of human development, probably in Egypt.

d) It is necessary to observe the rules of behaviour in public places.

e) Vehicles are permitted to pass when the traffic lights are at green.

f) It is impossible to build a perpetual motion machine.

g) Never bother another with something you can do yourself.

8. Are the following sentences judgements?

a) In what year was Alexei Tolstoy born?

b) Bring me the book on Wednesday evening.

Chapter IV

THE FUNDAMENTAL LAWS (PRINCIPLES) OF CORRECT THOUGHT

§ 1. The Concept of Logical Laws

Materialist dialectics, worked out by Marx, Engels and Lenin is the most profound and comprehensive teaching on development. It is based on fundamental laws: laws of the mutual transformation of quantitative and qualitative changes, the law of the unity and struggle of opposites and the law of negation of negation. The laws of materialist dialectics are universal: they are at work in nature, society and thought. Apart from the laws of materialist dialectics, there are many other laws at work in the objective world, which are the object of study by concrete sciences (physics, chemistry, biology, etc.). There also exist general scientific laws (e. g. the law of the conservation of energy).

A *law* is a necessary, substantive, stable and repetitive relation between phenomena.

Proceeding from this general definition of the category "law", we can also define the category "law of thought".

A *law of thought* is a necessary, substantive, stable and repetitive relation between thoughts.

The most simple and essential links between thoughts are expressed in the basic laws of formal logic. They include the law of identity, of noncontradiction, the law of the excluded middle and that of sufficient reason.

These laws are fundamental because they play an especially important role in logic, are the most general and form the basis for various logical operations with concepts and judgements, as well as being employed in the process of inference and demonstration. The first three laws were revealed and formulated by Aristotle. The law of sufficient reason was formulated by Leibniz.

The fundamental laws of logic are the reflection in human consciousness of certain relations between the objects of the objective world. Lenin repeatedly pointed out that the laws and categories of logic were reflections of the laws of development of nature and society, that the laws of logic were the reflection of the objective in subjective human consciousness.

The laws of formal logic may not be cancelled or replaced by others. They are of a universal human character, being the same for people of all races, nations, classes and occupations.

The laws of formal logic took shape as a result of human cognition in reflecting such typical properties of things as their stability, definitiveness and the incompatibility of the simultaneous presence and absence in one and the same object of the same properties.

The laws of logic are laws of correct thought and not laws referring to the actual things and phenomena of the world.

Apart from the four fundamental laws of formal logic, which reflect important properties of correct thought like definitiveness, non-contradiction, validity and clarity, as well as the choice of "either ... or" in certain "rigorous" situations, there are many non-fundamental laws of formal logic to which correct thought should conform in the process of working with its basic forms (concepts, judgements, inferences).

The laws of logic, be they fundamental or not, function in the thinking process as principles of correct reasoning in the course of proving true judgements and theories and disproving false judgements.

In mathematical logic, the laws expressed as formulas function as identically true statements. This means that the formulas expressing the laws of logic are true for any values of their variables. Of the identically true formulas, a special category is formed by those which contain one variable:

$\left.\begin{matrix} a \equiv a \\ a \rightarrow a \end{matrix}\right\}$ laws of identity.

$a \wedge \bar{a}$ – law of non-contradiction.
$a \vee \bar{a}$ – law of the excluded middle.

The connection between the logical criteria of the truth of knowledge and practical human activity

The criterion of practice does not act directly in all sciences. Here, one has to bear in mind the complex and mediate way in which reality is reflected in logical systems and their operations, logical forms and laws.

What is the correlation between the criterion of practice and the logical criterion of the truth of a conclusion in the context of an inference? On this score, Lenin wrote: "THE PRACTICAL ACTIVITY OF MAN HAD TO LEAD HIS CONSCIOUSNESS TO THE REPETITION OF THE VARIOUS LOGICAL FIGURES THOUSANDS OF MILLIONS OF TIMES *IN ORDER THAT* THESE FIGURES *COULD* OBTAIN THE SIGNIFICANCE OF *AXIOMS*."[1] This is why, in order to check the truth of conclusions in inferences, there is no need to resort to practice every time, but it is sufficient to use the logical (i. e., referring to the form of reasoning) criterion.

Practice is the decisive criterion of truth. The logical criterion of truth is auxiliary and derivative, stemming from practice and relying on practice as the ultimate criterion of truth.

In the history of philosophy and logic there have been repeated attempts to absolutise the logical criterion of truth. It is known that even Leibniz dreamed of a time when, instead of arguing, opponents would take their pens, sit down at a table and calculate. He attempted to devise a system of arithmetical logical calculus in the form of a calculating machine (algorithm) in order to then carry out calculations in accordance with formulated rules to determine what was true and what was false (without referring to practice, of course). However, Leibniz's idea that human thought could be completely replaced by a calculating machine was rather wishful thinking. It is impossible in principle to devise a system of calculus in order to always differentiate what is true from what is false.

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[1] V. I. Lenin, "Conspectus of Hegel's Book *The Science of Logic*", *Collected Works*, Vol. 38, p. 190.

§ 2. The Laws of Logic and Their Materialist Understanding

The law of identity

The law of identity is one of the laws of correct thought. The observance of this law guarantees that the thought process will be well defined and clear. The law is formulated as follows: "*In the process of reasoning, every concept and judgement must be used in the same meaning.*" The law of identity is written as "*a* is *a*" (for judgements) and "*A* is *A*" (for concepts), where *a* is used to denote any judgement, any thought, and *A* any concept. In mathematical logic, the law of identity is expressed in the following formulas:

$a \equiv a$ or $a \rightarrow a$ (in propositional logic) and

$A \equiv A$ (in the logic of classes, where classes are identical with the extensions of concepts).

Identity is the equality, the similarity of objects in some respect. For example, all liquids conduct heat and are resilient. Every object is identical with itself. In objective reality, identity exists in conjunction with difference. There are no two identical things, nor can there be (for example, two leaves of a tree, twins, etc.). Yesterday and today, one and the same thing is identical and different. For example, the outward appearance of a human being changes over the course of time, but we recognise and consider him as one and the same person. Abstract and absolute identity does not exist in reality: it would mean the end of development. But given certain conditions (within certain limits) we can disregard existing differences and fix our attention entirely on the identity of objects or their properties.

In thought, the law of identity functions as a normative rule. It denotes that in the process of reasoning one may not replace one thought with another or one concept with another. It is inadmissible to present identical thoughts as being different or different ones as being identical.

For example, the following three concepts are identical: Mikhail Lomonosov; founder of Moscow University; Russian scientist who discovered the law of the conservation and transformation of substances, since they denote one and the same person but give different information about him.

In thought, the law of identity is violated when someone speaks on a subject other than that which is being discussed, arbitrarily substitutes one subject of discussion for another and uses terms in something other than the accepted meaning without any forewarning. For example, an idealist is sometimes considered to be a person who believes in ideals and lives for the sake of some noble aim, whilst a materialist may be viewed as someone of a mercantile character who seeks profit and personal enrichment, etc.

It happens in the course of a discussion that it essentially becomes an argument about words. Sometimes people actually talk about different things, but think that they are referring to one and the same thing or event.

Logical errors are often made with the use of homonyms, i. e., words with two meanings ("minister", "material", "bill", etc.). Sometimes an error arises when using the personal pronouns "she", "it", "we", etc., when it is necessary to specify who or what is meant by the pronoun. The identification of different concepts leads to a logical error called *concept substitution*.

If the law of identity is violated, another error results which is called *thesis substitution*. In the process of proof or disproof, the thesis advanced is often deliberately or unconsciously substituted by another. In scientific and other discussions, this happens when something is ascribed to an opponent which he never actually said. This way of conducting discussions is inadmissible.

Identification is widely used in criminal investigations, for example in identifying objects, people, comparing handwritings, documents, signatures, fingerprints.

The law of identity is employed in science, art, data processing, school instruction and everyday life.

Concepts like "one", "two", "three", etc, are associated with an ability to distinguish and identify things, an ability which in both historical and logical terms precedes the ability to count them. The law of identity "*a* is *a*" (*a* is identical to *a*) was attributed to logic from the very earliest times. It is a natural thing to consider the predicate of identity a logical phenomenon. From the logical viewpoint, the concepts of identity and difference are logical constants.

In reality, absolute identity does not exist in changing objects. But in order to depict movement in thought, we have to resort to an idealisation and a simplification of reality. "The representation of movement by means of thought always makes coarse, kills,–and not only by means of thought, but also by sense-perception, and not only of movement, but *every* concept.

"And in that lies the *essence* of dialectics."[1]

In the sciences there exist various types and modifications of identity. For example, in mathematics we find equality, equivalence (equal cardinality, equality in numbers) of sets, congruence, identical transformation, identical substitution, etc.; in the theory of algorithms we find the identity of letters established by the abstraction of identification, the equality of alphabets ($A \equiv B$), the equality of specific words, etc.

Equalities are reflexive ($a \equiv a$) symmetrical (if $a \equiv b$, then $b \equiv a$) and transitive (if $a \equiv b$ and $b \equiv c$, then $a \equiv c$). The rule of replacing an equal with an equal can be applied to equalities.

There are also various types and modifications of difference: inequality, non-equivalence of sets, etc; in the theory of algorithms there exists the difference of letters, the inequality of concrete words (e. g. of an empty and a non-empty word), etc.

The law of non-contradiction

The objective world is such that it is impossible for one and the same object to display the presence and absence of the same properties at the same time.

For this reason, if object A has a certain property, then in their judgements about A people should affirm this property and not deny it. If someone affirms something and denies the same, or affirms something which is incompatible with what he has previously affirmed, then we have a logical contradiction. Contra-

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[1] V. I. Lenin, "Conspectus of Hegel's book *Lectures on the History of Philosophy*", *Collected Works,* Vol. 38, pp. 257–58.

dictions in formal logic are those arising from confused and incorrect reasoning. Such contradictions are an obstacle to the cognition of the world.

One should not confuse formal logical contradictions with dialectical ones. The law of the unity and struggle of opposites is universal, so that dialectical contradictions are a feature of nature, society and thought. It is a dialectical contradiction of life and not a contradiction of incorrect reasoning.

The Ancient Greek philosopher and scholar Aristotle considered the following to be the most certain of all principles: "It is impossible for the same attribute at once to belong and not belong to the same thing and in the same relation."[1] In this way, Aristotle gave the logical formulation of the law of non-contradiction: "No demonstration makes use of the principle that simultaneous assertion and negation are impossible..."[2] The formulation indicates that it is inadmissible for man to use formally contradictory statements in his thought or speach, since otherwise his thought will be incorrect.

A thought is contradictory if at the same time we assert and deny something about one and the same thing in one and the same relation. For example, "The Kama is a tributary of the Volga" and "The Kama is not a tributary of the Volga". Or "Lev Tolstoy was the author of the novel *Resurrection*" and "Lev Tolstoy was not the author of the novel *Resurrection*".

There will be no contradiction if we refer to different objects or to one and the same object taken at different times or in different relations. There will be no contradiction if we say "Rain is beneficial to mushrooms in autumn" and "Rain is not beneficial to the harvest in autumn", or "Sasha Golubev took first place in the table tennis competition" and "Sasha Golubev did not take first place in the running competition", since the objects of the thought in these judgements are taken in different

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[1] Aristotle, *Metaphysics*, Book IV, William Heinemann Ltd., London, 1980, p. 161.

[2] Aristotle, *Posterior Analytics*, I, William Heinemann Ltd., London, 1980, p. 75.

relations. The judgements, "Sasha Golubev is not one of the best runners" and "Sasha Golubev is one of the best runners" will not be contradictory if they refer to different times and will be contradictory if they refer to one and the same time.

The following four types of simple judgements cannot be true at the same time:

1. "The said *S* is *P*" and "The said *S* is not *P*".
2. "No *S* is *P*" and "All *S* are *P*".
3. "All *S* are *P*" and "Some *S* are not *P*".
4. "No *S* is *P*" and "Some *S* are *P*".

The second pair of judgements is such that both may be false, e.g. "No students are athletes" and "All students are athletes".

Contradiction in formal logic is most often defined as the conjunction of a judgement and its negation (*a* and *not-a*). But logical contradiction may also be expressed without negation; it occurs between two incompatible affirmative judgements. It is this type of logical contradiction which obtained between the judgements advanced, on the one hand by representatives of empirio-criticism, and the scientific assertions put forward by naturalists, on the other. Referring to the representatives of empirio-criticism, Lenin remarked: "...for the eclectic everything is 'compatible'!",[1] and he described the philosophy of Mach and Avenarius as follows: "We have seen that their philosophy is a hash, a pot-pourri of contradictory and disconnected epistemological propositions".[2]

Logical contradictions have social as well as gnoseological roots. In a letter to Proudhon, Marx wrote that the petty bourgeois were composed of "on the one hand" and "on the other hand". They were an embodiment of contradiction.

The examples given show that formal logical contradiction arises when an attempt is made to consider true two or more affirmative judgements which are in-

[1] V. I. Lenin, "Materialism and Empirio-Criticism", *Collected Works*, Vol. 14, 1962, p. 95.
[2] *Ibid.*, p. 217.

compatible. No less widespread is the form of logical contradiction whereby one and the same judgement is affirmed and negated at the same time, i. e., a conjunction is formed between *a* and *not-a*. In traditional formal logic, a contradiction is said to occur when two opposing (contrary or contradictory) judgements are made about one and the same object taken at one and the same time and in one and the same relation. In the calculus of statements used in two-valued logic, the law of non-contradiction is written as the formula $a \wedge \bar{a}$.

The law of non-contradiction reads as follows: "*Two opposing judgements may not be true at one and the same time and in one and the same relation*". Opposing judgements include: 1) opposite (contrary) judgements *A* and *E,* which may both be false; so they do not negate each other and may not be denoted as *a* and $\bar{a}$; 2) contradictory judgements *A* and *O*, *E* and *I* and such individual judgements as "This *S* is *P*" and "This *S* is not *P*", which do negate each other, since if one of them is true, the other must be false, and are therefore denoted as *a* and $\bar{a}$.

The formula used to express the law of non-contradiction in two-valued logic reflects only part of Aristotle's substantive law of non-contradiction, since it only relates to contradictory judgements (*a* and *not-a*) and does not extend to opposing (contrary) judgements. The formula $a \wedge \bar{a}$ is therefore insufficient and does not represent the substantive law of non-contradiction in its entirety. But we shall abide by tradition and call the formula $a \wedge \bar{a}$ the "law of non-contradiction", although the term is considerably broader than the formula.

If a formal logical contradiction is found in human thought (and speech), this thought is considered incorrect and the judgement from which the contradiction is derived is negated and considered false. When it comes to disproving the view of an opponent in polemics, frequent resort is therefore made to *reductio ad absurdum.*

Lenin often used this method. In a manner of speaking, he extended his opponent's arguments, continuing the chain of reasoning and deriving the logical consequences until the erroneous, false and absurd nature of these consequences became obvious. In his work *Materialism and Empirio-Criticism*, Lenin criticised the attempts made by Mach to find a third path in philosophy (to overcome the one-sidedness of both materialism and idealism) by introducing the idea that the world is a "complex of sensations" on the part of the subject, in this case the author of the said philosophical theory. This makes all objects, including other people, complexes of sensations. From this there logically follows the extreme solipsism according to which only *I* exist, the subject, and everything else is a complex of my sensations.

Dialectical contradictions in the process of cognition are sometimes expressed in the form of formal logical contradictions, e. g. the speeches made by a rapporteur and his opponent; by the counsel for the prosecution and that for the defence; views by people who ascribe to competing hypotheses; arguments by a doctor (or doctors on a panel) having obtained clinical analyses which are incompatible with the previous diagnosis of the illness, and many more besides. In all these and similar situations, the focus is on the incompatibility of judgements *a* and *not-a*, for example the incompatibility of any judgement *a* derived from a previous theory and judgement *not-a*, expressing the meaning of a new empirically obtained fact, i. e., the thought is established that the judgements *a* and *not-a* cannot both be true, and this means that their conjunction is false.

Here, the primary (content) is dialectical contradiction which emerges objectively in the process of cognition and serves as its motive force; the secondary is the means of fixation (expression) of dialectical contradiction in the form of the conjunction of two judgements *a* and *not-a*, i. e., as a formal logical contradiction.

We have a situation which, in terms of type, is similar to the "antinomical problem" case, when the dialectical contradiction which has emerged in cognition is expressed up to the moment of its solution as "*a* and *not-a*", i. e., it takes on the appearance, cover, external form of a formal logical contradiction, but essentially remains dialectical and needs to be solved through an investigation of the problem which has emerged. As a result of the dialectical synthesis of a thesis and an antithesis new knowledge is obtained which differs from both and does not represent their conjunction either. Thus, in thought a dialectical contradiction up to the moment of its solution sometimes takes on the form (structure) of a formal logical contradiction, and the revelation of the latter indicates the need for continued analysis and investigation of the situation which has emerged in cognition. The solution of this dialectical contradiction furthers the process of cognition.

A classical example of an antinomical problem is the well-known formulation of a cognitive problem by Marx in the first volume of *Capital*: "It is therefore impossible for capital to be produced by circulation, and it is equally impossible for it to originate apart from circulation. It must have its origin both in circulation and yet not in circulation."[1] In solving this problem, Marx reveals the secret of the emergence of surplus value: the main link on the road to this revelation was the establishment of the fact that capital arises in production but with the direct involvement of circulation.

The law of the excluded middle

Lenin repeatedly stressed the unity of the content of the laws of the objective world and the laws of thought: "The most common logical 'figures' ... are the most common relations of things."[2] Engels also noted that "laws of thought and laws of nature are necessarily in agreement with one another, if only they are correctly known".[3] These statements by Engels and Lenin on the objective nature of the laws of logic also refer to the formal logical law of the excluded middle. The ontological analogy to this law is the fact that a certain feature is either present in an object or not.

In his book *The Metaphysics*, Aristotle formulated the law of the excluded middle as follows: "Nor indeed can there be any intermediate between contrary statements,

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[1] Karl Marx, *Capital*, Vol. I, p. 163.

[2] V. I. Lenin, "Conspectus of Hegel's Book *The Science of Logic*", *Collected Works*, Vol. 38, p. 178.

[3] F. Engels, *Dialectics of Nature*, Progress Publishers, Moscow, 1972, p. 225.

but of one thing we must either assert or deny one thing, whatever it may be."[1]

The law of the excluded middle is based on the fact that a judgement can only have one of two truth values, the value "true" or the value "false". Aristotle proceeded from this to devise seven arguments convincingly proving that the negation of the law of the excluded middle is impossible.

In two-valued logic, the law of the excluded middle is formulated as follows: "*Of two contradictory judgements, one is true, the other is false, and a middle value does not exist.*" We give the name contradictory to two judgements, one of which affirms something about an object and the other denies that same thing, since they cannot both be true or false at the same time. One of them is true and the other must necessarily be false. Such judgements are said to be mutually negating. If we denote one of a pair of contradictory judgements by the variable a, we should denote the other by the variable $\bar{a}$.

The following pairs of judgements are mutually negating:

1. "This S is P" and "This S is not P" (individual judgements).
2. "All S are P" and "Some S are not P" (judgements A and O).
3. "No S is P" and "Some S are P" (judgements E and I).

In respect to contradictory judgements (A and O, E and I), both the law of the excluded middle and the law of non-contradiction apply, this being one of the points of convergence of the two laws.

The difference in the areas of application of these laws is to be found in the fact that, in relation to the opposed (contrary) judgements A and E (e. g. "All mushrooms are edible" and "No mushrooms are edible"), which cannot both be true, but can both be false, the law of non-contradiction applies whereas the law of the excluded middle does not. This means that the field of application of the substantive law of non-contradiction is broader than the field of application of the law of the

[1] Aristotle. *The Metaphysics,* Book IV, pp. 199–201.

excluded middle (which holds true only for contradictory judgements, i. e., judgements of the type *a* and *not-a*). Indeed, only one of the following judgements is true: "All houses in the said village have electricity", "Some houses in the said village don't have electricity"; no intermediate value is possible.

In classical two-valued logic, the law of the excluded middle is expressed by the formula $a \vee \bar{a}$ (the sign "$\vee$" denotes inclusive disjunction, the connective "or"). It would be more precise to express this law as $a \dot{\vee} \bar{a}$, where "$\dot{\vee}$" denotes exclusive disjunction, characterising the incompatibility of a and $\bar{a}$. But here and in the following we shall abide by the widely accepted formula for this law: $a \vee \bar{a}$.

In both its substantive and formalised forms, the law of the excluded middle refers to one and the same range of judgements – contradictory ones, i. e., judgements that negate each other.

It is not possible to derive Aristotle's substantive laws of non-contradiction and the excluded middle from each other, since the fields of definition to which they are applicable are different.

Given the fact that in the formalised laws of non-contradiction and the excluded middle, i. e., in the formulas $\overline{a \wedge \bar{a}}$ and $a \vee \bar{a}$, the areas of definition of the propositional variables (i. e., the variables expressing a judgement and its negation: a and $\bar{a}$) are the same (only contradictory judgements are included), it is possible on the basis of De Morgan's law [i. e., the formula $\overline{a \wedge b} \equiv \equiv \bar{a} \vee \bar{b}$, the law of cancellation of double negation, i. e., $\bar{\bar{a}} \equiv a$ and the law of commutative disjunction, i. e., the formula $(a \vee b) \equiv (b \vee a)$], by means of elementary equivalent transformations to derive the law of the excluded middle from the law of non-contradiction (and vice versa):

$$\overline{a \wedge \bar{a}} \equiv \bar{a} \vee \bar{\bar{a}} \equiv \bar{a} \vee a \equiv a \vee \bar{a}.$$

In thought, the law of the excluded middle presupposes the clear choice of one of two mutually exclusive alternatives. This demand must be fulfilled if a discussion is to be conducted in a correct manner.

The specific nature of the action of the law of the excluded middle in the presence of cognitive "indeterminacy".

As already pointed out, it is an objective condition for the laws of non-contradiction and the excluded middle to operate in thought that there must be in nature, society and thought itself relatively stable states of objects, consistency and definitiveness of properties and relations between objects.

But changes occur in nature and society, objects and their properties undergo transition to become the opposite. For this reason, transitional states and intermediate situations are no rare occurrence. Indeterminacy in actual cognition [in one of its forms (stages)–abstract thought] results, first, from the reflection of "transitional" states of actual objects in reality and, second, from incompleteness, inaccuracy (at some stage of cognition), not entirely sufficient reflection of the object of cognition in the course of its study.

Let us analyse several "transitional" situations encountered in nature, society and cognition. In nature, the instability of the movement of air currents carrying cyclones and anticyclones gives rise to frequent changes in the weather. Uncontrollable natural disasters like earthquakes, floods, volcanic eruptions, droughts and torrential rain bring human suffering in their wake. It is still not always possible to precisely forecast an earthquake, flood or many other natural occurrences, and this "indeterminacy" in our cognition means that people are not able to prepare in good time for these undesirable natural occurrences.

In line with a tradition dating back to Aristotle, some experts on logic consider that in situations referring to the future the law of the excluded middle is inapplicable, since the statements, "A sea-fight must take place on the morrow" and "A sea-fight must not take place on the morrow" are neither true nor false today, but both are indeterminate. Indeed we are not able to say which of the two contradictory judgements: "There will be an earthquake in Tashkent in a month's time" and "There will be no earthquake in Tashkent in a month's time" is true and which is false. At the same time, we are able to predict a solar eclipse centuries in advance to an accuracy of one second, since in this rigorous situation the law of the excluded middle acts unlimitedly. Thus we are able to say which of the following is true and which is false: "There will be a solar eclipse in Moscow on 27 December 2088" and "There will not be a solar eclipse in Moscow on 27 December 2088" – even though both of these judgements refer to the future. Therefore, the possibility of applying the law of the excluded middle to individual future events should be examined on each occasion according to the concrete statements.

In both society and nature, indeterminate situations, transitional periods and states exist alongside determinacy and stability. There exist unpredictable, accidental incidents, such as air crashes, railway and road accidents, etc. It is as a rule impossible to predict an individual disaster, so that one cannot apply the law of the excluded middle to this situation. One might object that the law of the excluded middle only means that one of the contradictory judgements is true and the other is false, and no intermediate possibility exists, and it is a task of concrete analysis to decide which judgement is true. However, it is impossible to carry out a concrete analysis of future events and say with accuracy whether a certain aircraft will land successfully or not, whether an aircraft sent on a combat mission will return to base or not. None of these judgements has a determinate truth value.

For this reason, it is only possible to determine the truth of one of two contradictory judgements referring to individual (concrete) future events with a certain degree of probability (likelihood). People behave like this in practice, more or less hoping for success and thus assessing the degree of likelihood, the degree of truth of a given judgement.

Indeterminate situations often crop up in cognition, and not only because such situations occur in nature and society, or because the process of cognition has not been completed, but also because of the need to introduce a third truth value, "indeterminate", into the actual processes of investigation, cognition and training. For example, in sociological questionnaires distributed with a view to studying public opinion, allowance is made in advance for the uncertainty of the response, because, first, a person may answer "Don't know" or give no answer at all. When the data from sociological studies are processed on a computer, the program employed should not only take account of definite answers "yes" and "no" but also of indefinite ones. In the process of programmed instruction using teaching machines, the answers to the questions are divided into these groups: 1) "true reply" (or solution), 2) "false reply" (or solution), 3) "don't know". Thus, when students' knowledge is checked by means of a machine, a third truth value ("indeterminate") is introduced from the outset, and the law of the excluded middle does not apply.

In science and everyday practice, people often have to analyse concepts having the qualities of flexibility and mobility and with no "rigorously" fixed extension (for example, the concept "young man", "old man", "fashionable dress" and many others besides).

Mathematics, logic, cybernetics and other sciences make use of concepts with "rigorously" fixed extensions and algorithms precisely setting out the sequence of operations with these concepts. But in the process of reflecting objective reality, we have to work in our minds with flexible concepts as well, to encounter what are called diffuse algorithms and to cope with methods allowing the resolution of problems which entail indeterminacy by the very way they are posed. In the theory of "diffuse" sets, which makes use of such concepts, the law of the excluded middle and the law of non-contradiction are not applicable.

The above examples describe situations where the law of the excluded middle is inapplicable or applicable only to a limited extent: in a certain field or at a certain stage of cognition. Let us examine situations in which the law of the excluded middle is partially applicable.

In the process of voting, it is permissible to vote for the adoption of a resolution in accordance with a three-valued system of logic: "for", "against", "abstain", and here the law of the excluded middle is inapplicable. However, the votes are counted in accordance with a two-valued system of logic: either the resolution will be adopted or not, there is no intermediate possibility.

Marx wrote: "In *jus* [law.– *Ed.*] it must be either-or."[1] Indeed, in judicial practice it is necessary to prove the judgement that the said fact (crime) took place, or to disprove it, and there is no third possibility. When an appeal is lodged, a superior court again takes a decision according to the law of the excluded middle: "either guilty or not guilty, with no intermediate possibility". But as long as the proceedings have not been completed, the judgement, say, "This man is guilty of arson" is not yet proven and not yet disproven and neither true nor false, but indeterminate.

Logical laws should be applied in a concrete manner, depending on the properties of the fields in which they are used, and this fully applies to the law of non-contradiction and the law of the excluded middle.

In cognition, there frequently emerge indeterminate situations reflecting transitional stages present both in material phenomena and the process of cognition itself (for example, the state of clinical death; the situation when a hypothesis has not been proved and not yet disproved; when we do not know the degree of accuracy of a long-term weather forecast; arguments about individual future events, etc.). In such situations, we cannot just think according to the laws of classical two-valued logic, and resort to three-valued logic in which judgements may take on one of three truth values: truth, falsehood or indeterminacy. In some of these multi-valued logics, the law of non-contradiction is not an invariably true formula.

Thus, the law of the excluded middle applies when cognition is dealing with a rigorous situation: either-or, truth-falsehood; the law of the excluded middle cannot be applied where we are concerned with a reflection of indeterminacy in objective processes or the process of cognition itself. This therefore calls for a concrete analysis of the concrete situation, taking into account the peculiarities of the object field.

The law of sufficient reason is formulated as follows: "Every true thought should be sufficiently substantia-

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[1] Karl Marx, "Herr Vogt" in K. Marx, F. Engels, *Collected Works*, Vol. 17, Progress Publishers, Moscow, 1981, p. 284.

ted". This refers only to a substantiation of true thoughts, since it is impossible to substantiate thoughts which are false. There is a Latin proverb which says, "to err is human, but to insist on one's error is foolish".

There is no formula to express this law, since it is only of a substantive nature. Books sometimes give the formula $a \to b$ to express this law. However, in two-valued logic there exist paradoxes of material implication due to the fact that the formula $a \to b$ is true even if a and b are false or a is true and b false. For example, in propositional calculus in two-valued logic, the following judgement is considered true: "If a tiger is a herbivorous animal (a), then five times six is forty (b)"–although in the practice of correct thought and in natural language such a statement is not only false but has no sense, since there is no meaningful connection between the simple judgements (a) and (b) and, moreover, judgement (b) does not follow from judgement (a). Since logical material implication, expressed in the formula $a \to b$, and the substantive conjunction "if... then" do not entirely coincide, the law of sufficient reason cannot be expressed by the formula $a \to b$.

As arguments to substantiate true thoughts, it is possible to use true judgements, statistics, scientific laws, axioms and theorems.

Logical reason and logical consequence do not always coincide with actual cause and effect. For example, rain is the actual reason why rooftops become wet. The logical reason and consequence will be exactly the opposite, since, having looked out of the window and seen the wet rooftops (logical reason), we derive the logical consequence: "It has been raining". The conclusions reached by Sir Arthur Conan Doyle's literary hero Sherlock Holmes are quite striking. He used the effect to arrive at the cause by inferring with a high degree of certainty the logical consequence, i.e., the actual cause of the event from the logical reason, i.e., the actual effect. In making a diagnosis, doctors also proceed from the actual effect to find the actual cause, which means that their conclusions need to be checked with particular thoroughness and backed up with sound arguments.

Special demonstrative power is characteristic of arguments in scientific research and in the teaching process, when it is inadmissible to take unproven assertions for the truth.

The principles of demonstration, ways and means of substantiating true thoughts and disproving false ones will be dealt with in greater detail in the chapter on the logical foundations of the theory of argumentation.

§ 3. The Use of Formal Logical Laws in Cognition and The Teaching Process

The laws of formal logic apply to cognition in whatever form, but in teaching it is especially important to make conscious use of them, since teaching is geared to forming pupils' correct thought. When used in such a way, the laws of formal logic function as normative rules of thought. The *law of identity* as a normative rule of thought, forbid the substitution of any concept (or judgement) with another concept (or judgement) in the process of reasoning and rules out the use of terms in different senses, but demands the precision, clarity and unambiguity of concepts. In teaching work, this manifests itself in the need to clearly define the concepts introduced. In the learning process, pupils encounter synonyms (disease-illness, mistake-error) and homonyms (class, group, etc.). The use of homonyms is particularly hazardous when they have closely related meanings. In teaching, it is essential to avoid homonyms, since every term and every sign (symbol) should be defined unambiguously. In mathematics, errors sometimes occur due to the use of one and the same term in different senses. For example, the notation $[AB]$ used to mean both a segment with ends A and B and its length; now the segment is denoted by $[AB]$ and its length by $|AB|$, so that $|AB| = 3$ cm is read as "the length of section AB is equal to 3 cm". The word "figure" was used to denote numbers between 0 and 9, which led to confusion in the presentation of material.

In mathematics, the clarity and unambiguity of concepts and symbols employed demands a special mathematical language, one that is concise and accurate and has rules which, in contrast to the rules of grammar, allow no exceptions. "If we accept this viewpoint, setting up equations appears as a sort of translation, translation from ordinary language into the language of mathematical symbols."[1]

In analysing a new problem, pupils should introduce appropriate means of denotation. George Polya considers that a good system of denotation should satisfy the following demands: it should be unambiguous, substantive and easy to remember. It is inadmissible to use one and the same sign to denote different objects (in one and the same problem), but one can use various symbols for one and the same object (for example, a conjunction of judgements can be denoted as $a \& b$ or $a \wedge b$ or $a \cdot b$). The language of mathematical symbols facilitates the solution of mathematical problems.

No less important is the use of the law of identity in the study of one's native or a foreign language, literature, history, sociology, etc. As in mathematics, the law of identity calls for the unambiguous use of concepts and rules out the logical error of "concept substitution".

In the study of literature, the teacher uses the law of identity in teaching pupils how to write essays. A violation of the law of identity leads to digression from the subject under discussion, the substitution of one subject for another. In writing essays, an ability is called for to recognise the confines of the subject under discussion, to select the relevant material, to develop and prove the fundamental thought behind the essay. Shortcomings in essays manifest themselves in a violation of correct composition (absence of an introduction, conclusions on the theme under discussion, long-windedness, violation of narrative logic). The laws of logic (including the law of identity) demand clarity, brevity, an ability to fully cover the subject of the essay, consistency and the correct construction of the system of argumentation. However, some pupils narrow down the subject, being unable to make generalisations and draw conclusions, to find the appropriate word from their own language. Some pupils answer the questions and give renderings of what they have read in "bookish"

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[1] George Polya, *How to Solve It*, Princeton University Press, 1946, p. 124.

phrases being unable to give the basic idea in their "own" words (this also applies to translations from a foreign language).

The law of identity is used in the execution of dividing operations and also in constructing various classifications based on a constant property. When the principle of constancy is violated a logical error results, with the elements of division not being mutually exclusive.

The law of identity provides the basis for *identification*, an operation widely employed by criminologists, historians (in the course of studying archeological finds), linguists, biologists, chemists, geologists, geographers, etc. In teaching their given disciplines, lecturers make use of the necessary material to substantiate the identification of different objects in the course of their study. Correct identification provides us with knowledge about the common features of objects.

Associated with the law of identity is the *law of non-contradiction*, since the former expresses a relation of logical identity and the latter, a relation of logical incompatibility. The use of the laws of identity and non-contradiction is closely connected with the operation of comparison in the course of which similarities and differences are established between the objects being examined. In comparison we encounter two forms of incompatibility: a and $\bar{a}$ (the first, simpler form); a and b, where b is split into *not-a* and c (the second, more complex form). The law of non-contradiction covers both these forms of incompatibility. When applied to judgements, the form a and $\bar{a}$ expresses the relations between judgements A and O as well as E and I (see "logical square"). The form a and b expresses the relations between judgements A and E.

In cognition, the law of non-contradiction is used in the dichotomous division of concepts, where the concept A is divisible into B and *not-B* (e.g. plants are divided into edible and inedible). B and *not-B* are incompatible concepts and in contradiction to each other (i.e., they are contradictory concepts). Opposite concepts are also incompatible (white paper – black paper; reprimand – award; hope – despair). Like the law of identity, the law of non-contradiction is applicable not only to judgements, but also to concepts and, in the logic of classes, to classes, where it is expressed by the formula $\overline{A \cdot \bar{A}}$ [the letter A denotes a class (set)]. When we add a complement to class A, denoted as A', for which the law $A \cdot A' = \emptyset$ applies (the intersection between class A and its complement gives an empty set), this is just another way of expressing the law of non-contradiction in a form that is applicable to concepts but not to judgements.

In respect of concepts, the law of non-contradiction operates in the use of antonyms in written and oral language, that is words which are opposite in terms of their basic meaning and denote the opposition of certain objects, qualities, actions, states, phenomena, desires, results, etc. (e. g. giant – dwarf, prolongation – shortening, harmony – disharmony, symmetry – asymmetry, hard work – easy work, etc.).

Depending on the *type of opposition* being expressed, antonyms are divided into the following classes: 1. *Antonyms expressing qualitative opposition.* "Complete and true antonymy is expressed by the extreme symmetrical elements of such opposition; those in between indicate an increment (or loss) in the degree of quality: easy (simple, trifling), not difficult, of medium difficulty, not easy, difficult (complicated)". 2. *Antonyms expressing complementation.* These are a relatively small class of antonyms which are two opposing elements complementing each other to express a certain essence, so that the negation of one gives the value of the other: not + single = married (blind – sharp-eyed, finite – infini-

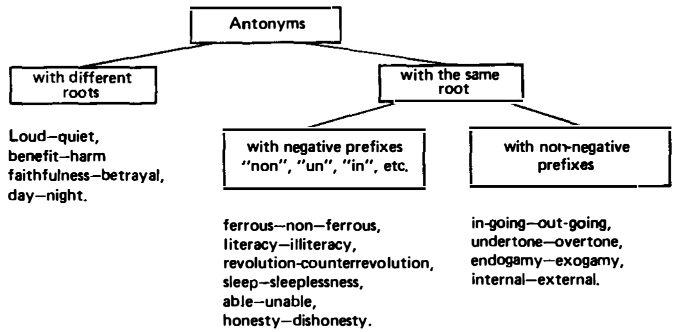

Fig. 49

te). 3. *Antonyms expressing the opposite direction of actions, features and properties* (dismantle – assemble, increase – reduce, light – extinguish, put out, etc).

Depending on the means of their formation, antonyms may be classified by means of dichtomous division (i. e., into *A* and *not-A*) in the following manner.

Antonyms may be expressed by formally varying means, so that one word may be opposed by two or even several. For example, for the word "friend" there are two antonyms, namely "enemy" and "foe".

Incompatible concepts which are contradictory or opposite may be expressed by antonyms having various structures: 1) *A – B* (good – evil, hero – coward); 2) *A – not-A* (literacy – illiteracy; ability – inability).

The law of non-contradiction applies to concepts of both types and thus to antonyms of the two types indicated.

In order to avoid violating the law of non-contradiction, one should make careful use of antonyms in both written and oral language. One should distinguish between the shades of meaning of two antonyms applying to the same word (e. g. action – inaction, action – counteraction; good – poor).

In literature seminars, students are familiarised with examples of contradiction in the thought of literary characters and learn to analyse contradictions in their own work and the answers given by fellow students.

If someone asserts something and then denies the same thing, i. e., contradicts himself, then his argument is false.

In Turgenev's novel *Rudin*, there is the following dialogue between Rudin and Pigasov:

> "Wonderful," said Rudin. "It seems you think there is no such thing as conviction."
> "Right, it doesn't exist."
> "Is that your conviction?"
> "Yes."
> "But you said there's no such thing. That is one example for you straightaway."
> Everyone in the room smiled and exchanged glances."

(I. S. Turgenev, *Rudin*, Foreign Languages Publishing House, Moscow, 1954, p. 31.)

In mathematics, frequent use is made of the method of reduction to the absurd (*reductio ad absurdum*). The use of this method is based on the law of non-contradiction: if a contradiction follows from assumption *a*, i.e., $(b \wedge \bar{b})$, then *a* should be rejected as erroneous. However, George Polya puts forward a number of arguments to substantiate the shortcomings of the *reductio ad absurdum* method and the method of indirect proof, since we are forced all the time to concentrate our attention not on a true theorem, which should be remembered, but on a false assumption, which should be forgotten. The verbal form of such argument, Polya points out, can become tiring and even unbearable due to the constant repetition of the words "hypothetically", "supposedly" and "allegedly".[1] However, it would be unwise to entirely reject *reductio ad absurdum* in mathematics, although it is better where possible to substitute this method of indirect proof with direct demonstration.

The law of non-contradiction is used in the conduct of debate. For example, the judgement put forward by one party and the contradictory judgement advanced by the other (e.g. *A* and *O*) cannot both be true at the same time and in one and the same relation; one of them must be false. In the course of the discussion, the falsehood of one of the judgements should be proven. Arguments are employed in the discussion of ethical, aesthetical and other issues. The subject of the discussion is a question which, both in literature and in life itself, different people solve in different ways. The problem being studied allows for various interpretations (for example, moral problems), and in the course of the discussion students arrive at a correct conclusion by comparing, analysing and discussing various points of view.

The *law of the excluded middle* is also used in the teaching process, and in many different situations at that: we shall just refer to some of the most important ones here. The law of the excluded middle calls for the choice of one of two mutually exclusive alternatives.

Similarly to the law of identity and the law of non-contradiction, the law of the excluded middle is applicable not only to judgements, but also to concepts (the formula $A \vee \bar{A}$ for classes). In accordance with this

[1] *Ibid.*, p. 153.

formula, the concept is dichotomously divided into two mutually exclusive and mutually complementary (to the point of universality) classes. Dichotomy is employed in all sciences and thus also in the teaching of any discipline. For example, sentences may be simple or compound (not simple); attention may be voluntary or involuntary, a series of numbers may be finite or infinite, etc., and there is no intermediate possibility apart from A and *not*-A.

The complement to class A, i.e., A', is constructed according to the law of the excluded middle and subject to the formula $A + A' = 1$. This formula and the construction of the complement to class A are used widely in mathematics.

In the teaching process, a major role accords to the *law of sufficient reason.* This is reflected in the demand for demonstrability in the presentation of material and for an optimal selection of information in students' answers. A separate chapter on the logical foundations of argumentation is devoted to this subject, and we refer readers to this part of the book.

Chapter V
INFERENCE

§ 1. The General Concept of Inference

The three forms of thought are concepts, judgements and inferences. We can use many different types of inferences to obtain new knowledge indirectly. It is possible to construe an inference if we have one or a number of related true judgements (called premises). Let us take an example of inference:

All carbons are flammable.
Diamonds are carbon.

Diamonds are flammable.

The structure of any inference contains premises, a conclusion and a logical relation between the premises and the conclusion. The logical transition from the premises to the conclusion is called inference. In the example given above, the first two judgements, standing above the line, are premises. The judgement "Diamonds are flammable" is the conclusion. In order to check the truth of the conclusion "Diamonds are flammable", there is no need to resort to immediate experience, i. e., to burn a diamond. The conclusion on the flammability of diamonds can be obtained with absolute certainty by relying on the premises being true and by following the rules of inference.

Inference is a form of thought which allows, by observing certain rules, to obtain a new judgement, either necessarily or with a certain degree of probability, from one or several judgements.

The process of drawing conclusions from the premises by the rules of deductive inference is called inference.

The concept of logical sequence

The derivation of corollaries from given premises is a widespread logical operation. It will be recalled that the truth of the conclusion depends on the truth of the premises and the validity of the inference. Sometimes, in converse inference deliberately false premises (the so-called antithesis in indirect proof), or non-proven premises are used. However, these premises have to be later discounted.

One who has not studied logic reaches these conclusions without consciously using the figures and rules of inference. Mathematical logic provides a formal apparatus by means of which it is possible, in certain divisions of logic, to derive corollaries from the given premises. By using this apparatus, we can, if we have a number of pieces of information, obtain new information from them which does not directly follow from our premises but is contained in them. We can thus derive the logical corollaries from the information in question.

A logical conclusion from the given premises is a proposition which cannot be false if the premises are true.

In other words, some expression B is said to be the logical conclusion from formula A (where A and B are used to denote two propositions differing in form), if, having substituted variables for the concrete elementary statements denoted by A and B, we obtain an identical true expression ($A \rightarrow B$), or a law of logic.

Let us take the following example. We have been given three premises: (1) "If John is the brother of Maria or John is the son of Maria, then John and Maria are relatives"; (2) "John and Maria are relatives"; (3) "John is not the son of Maria". Can we derive from this the logical conclusion, "John is the brother of Maria"? Many will believe at first sight that this logical conclusion from the three premises is true. In order to check it, we must construct the formula of the inference. Let us denote the judgement "John is the brother of Maria" by the letter (variable) a, the judgement "John is the son of Maria" by b, and the judgement "John and Maria are relatives" by c.

We may now write our problem in symbols (the three premises are written above the line and the supposed conclusion below it):

$$\frac{(a \dot{\vee} b) \rightarrow c,\ c,\ \bar{b}.}{a}$$

By joining up the three premises with the conjunction "$\wedge$" and adding the supposed conclusion a by means of "$\rightarrow$", we obtain the formula:

$$(((a \dot{\vee} b) \rightarrow c) \wedge c \wedge \bar{b}) \rightarrow a.$$

Let us check whether this formula, in which a, b and c are now interpreted as variables, is a law of logic.

For this purpose, let us compose a table for the said formula.

a	b	c	$\bar{b}$	$a \dot{\vee} b$	$(a \dot{\vee} b) \rightarrow c$	$((c \dot{\vee} b) \rightarrow c) \wedge c \wedge b$	$((a \dot{\vee} b) \rightarrow c) \wedge$ $\wedge c \wedge b) \rightarrow a$
T	T	T	F	F	T	F	T
T	T	F	F	F	T	F	T
T	F	T	T	T	T	T	T
T	F	F	T	T	F	F	T
F	T	T	F	T	T	F	T
F	T	F	F	T	F	F	T
F	F	T	T	F	T	T	F
F	F	F	T	F	T	F	T

In the last column, the formula is false in one instance, so it is not a law of logic. From our three premises, it therefore does not necessarily follow that "John is the brother of Maria". John could be the nephew of Maria, or the uncle of Maria, or some other relative.

This example shows that the effectiveness of mathematical logic is apparent when it is difficult to establish with formal logic whether a certain conclusion follows from the given premises or not, and particularly in cases where we have to deal with a large number of premises (but have no experience of dealing with formulas containing quantifiers).

Inferences may be *deductive, inductive* or *inferences by analogy*. The inference may be logically necessary, i. e., yield a true conclusion, or probable (likely), i. e., yield a conclusion which follows from the given premises only with a certain degree of probability.

A correctly construed deductive inference implies the necessary procedure of drawing the logical conclusion from the given premises.

§ 2. Deductive Inferences

A deductive inference is one in which the conclusion necessarily follows from the premises which express knowledge with a higher degree of generality, and itself represents an item of knowledge with a lesser degree of generality.

For example,

All fish breathe through their gills.
All perch are fish.

All perch breathe through their gills.

In this case, the first premise "All fish breathe through their gills" is a universal affirmative judgement and expresses a greater degree of generalisation than the conclusion, which is also a universal affirmative judgement: "All perch breathe through their gills". We construct an inference from the property of a genus ("fish") to its applicability to a species ("perch"), i.e., from a general class to a particular case, a subclass. A particular case should not be confused with a particular judgement "Some *S* are *P*" or "Some *S* are not *P*".

The concept of the rules of inference

An inference yields a true conclusion if the premises are true and the rules of inference are observed. The rules of inference, or rules of conversion of judgements, allow a transition to be made from premises (judgements) of a certain type to conclusions also of a certain type. For example, if we have as premises two judgements, "$a \dot{\vee} b$" and "$\bar{a}$", then, according to one of the rules of inference, it is possible to make a transition to judgement "b". This may be written as follows: $((a \dot{\vee} b) \wedge \bar{a}) \rightarrow b$. This formula is a law of logic.

Correct logical reasoning may concern any objects. Logical errors can also occur in arguments with any object content. It does not, of course, follow from this that one and the same apparatus of formal logical rules should be applicable to any object field. This apparatus should itself develop with the advance of science and human practical activity. One of the characteristic features of logic is that, once some information has been obtained about the circumstances of an issue, it allows the derivation – or, more precisely, revelation – of new knowledge contained in their totality. Thus, by observing the moon and the sun and drawing logical conclusions from the observations (including inductive generalisations), people were able to logically conclude from them, even as far back as ancient times, accurate forecasts about the occurrence of solar and lunar eclipses.

Another characteristic feature of logic, which is closely associated with the previous one, is that any logical conclusion from the premises presupposes some or other formalisation, i.e., it may be executed according to some or other general rules relating to the methods of expressing knowledge and the means of processing these expressions: the means of forming and converting expressions. Depending on the means we have at our disposal, there are many possible methods of formalisation, starting with the fact that one and the same piece of knowledge may be expressed in different languages. But some language or other ("language" does not necessarily mean its sound form) we have to use. Thought is impossible without language, without a material way of expressing it.

The formalisation of the methods of inference is connected above all with the fact that each step in this process is carried out only in accordance with one or other of the rules of inference listed earlier, which apply only to methods of working with formal expressions of thought by means of material signs. Among the latter there are those which are specific to logic, which are known as logical constants. In mathematical logic, they are conjunction, disjunction, negation, implication, equivalence, quantifiers of generality and existence, etc.

The choice of the rules of inference in a logical system is not an arbitrary matter. The rules of inference should satisfy a number of demands. First, they must allow only a true conclusion to be derived from the true premises.

Second, these rules must be *non-contradictory* (compatible) in the said logical system and such that it is not possible to derive formula a by one method and formula $\bar{a}$ by another, i.e. violate the law of non-contradiction. Otherwise, it would be possible to derive (prove) anything at all in a contradictory logical system, both a truth and a falsehood. Third, the system of the rules of inference must be *complete*, which means that we should be able to use only these rules in the logical system in question to derive any substantively true conclusion formulated in the terms of the logical system and following from the given premises.

A distinction is made between the *rules of direct inference* and the *rules of indirect inference*. The rules of direct inference allow a true conclusion to be drawn from the existing true premises. The rules of indirect inference allow judgements to be made about the truth of some conclusions from the truth of other conclusions (these rules will be analysed in §10 of this chapter).

Deductive inferences are construed on the basis of the rules of direct inference. There exist the following types of deductive inferences (conclusions): conclusions depending on the subject-predicate structure of judgements; conclusions based on logical relations between judgements (conclusions of propositional calculus).

We shall now go on to examine these types of conclusions.

Let us look at conclusions based on the subject-predicate structure of judgements.

The following conclusions from categorical judgements are among those typical in the practice of argumentation: (1) conclusions reached by converting judgements; (2) the categorical syllogism, the incomplete syllogism (enthymeme), complex (polysyllogisms) and complex incomplete syllogisms (sorites, epihairems).

§ 3. Conclusions from Categorical Judgements by Means of Their Conversion

Deductive conclusions reached from one premise are called *direct inferences*. They include the following: reduction, converse, predicate opposition, and the "logical square" inferences.

Reduction

A *reduction* is a type of direct inference in which the quality of the premise is changed without any change occurring in its quantity. As already pointed out, depending on the quality of the copula ("is" or "is not"), categorical judgements may be affirmative or negative.

Particular affirmative judgements are reduced to particular negative ones and vice versa, and universal affirmative judgements to universal negative ones, and vice versa.

Reduction is effected in two ways:

(a) by means of double negation of the copula and the predicate:

S is *P*. → *S* is not *not-P*.

Subjects are the principal elements of sentences. → No subject is not the principal element of a sentence;

(b) negation may be transferred from the predicate to the copula.

S is *not-P*. → *S* is not *P*.

All halogens are non-metals. → No halogens are metals.

Reduction may be applied to all four types of judgement, *A*, *E*, *I* and *O*.

1. *A* → *E*.

Structure: All *S* are *P*. → No *S* is not *not-P*.

All wolves are predators. → No wolf is not a predator.

2. *E* → *A*.

No *S* is *P*. → All *S* are *not-P*.

No polyhedron is a plane figure. → All polyhedrons are non-plane figures.

3. *I* → *O*.

Some *S* are *P*. → Some *S* are not *not-P*.

Some mushrooms are edible. → Some mushrooms are not inedible.

4. *O* → *I*.

Some *S* are not *P*. → Some *S* are *not-P*.

Some elements of a sentence are not principal ones. → Some elements of a sentence are non-principal ones.

Conversion

A *conversion* is a type of direct inference in which the subject and the predicate of the premise are transposited in the conclusion (the new judgement), i. e., the subject and predicate change places whilst the quality of the judgement is preserved.

Let us give four examples:

1. All dolphins are mammals. → Some mammals are dolphins.

2. All flat angles are angles whose sides form one straight line. → All angles whose sides form one straight line are flat angles.

3. Some pupils are stamp collectors. → Some stamp collectors are pupils.

4. Some musicians are violinists. → All violinists are musicians.

There are two types of conversion: *simple*, or *pure* (examples 2 and 3) or *limited* (examples 1 and 4).

A conversion will be pure, or simple, when *S* and *P* in the premise are both distributed or both undistributed. A limited conversion occurs when the subject of the premise is distributed and the predicate is undistributed, and vice versa, i. e., when *S* is undistributed and *P* is distributed.

Examples:

1. Judgement *A* is universal and affirmative.

(a) "In Euclidean geometry, all parallel lines are straight lines lying in the same plane and having no common points" (definition).

After conversion, the above judgement becomes the following: "All straight lines lying in the same plane and having no common points are parallel lines in Euclidean geometry". This is an example of pure or simple converse.

(b) Judgement *A* "All Soviet cosmonauts are Heroes of the Soviet Union" may be converted with a limitation: "Some Heroes of the Soviet Union are Soviet cosmonauts".

2. Judgement *E* is universal and negative.

Since *S* and *P* are always distributed in this case, its converse is pure or simple.

"No trapezium is an equilateral figure" → "No equilateral figure is a trapezium".

3. Judgement *I* is particular and affirmative.

There are two kinds of converse:

(a) The converse is real if *S* and *P* are both undistributed. For example, the judgement "Some plants are poisonous" may be converted into the following judgement, "Some poisonous organisms are plants".

(b) A limited converse obtains when the extension of *P* is less than the extension of *S*, i.e., *P* is distributed and *S* is not. For example, the judgement, "Some musicians are composers" is converted into "All composers are musicians".

4. Judgement *O* is particular and negative.

This judgement cannot be converted, since no necessary conclusions follow from judgement *O*. For example, it is impossible to obtain a true converse from the particular negative judgement "Some animals are not dogs".

Predicate opposition

A predicate opposition is a type of direct inference in which the subject of the new judgement (i.e., the conclusion) is a concept contradicting the predicate of the premise and the predicate is the subject of the premise; the copula is also changed to the opposite.

In other words, we proceed in the following way: (1) instead of *P* we take *not-P*; (2) we transposit *S* and *not-P*; (3) we change the copula to the opposite.

For example, we have the judgement: "All lions are predators". By way of predicate opposition, we obtain the judgement: "Any non-predator is not a lion."

Predicate opposition may be considered the result of two consecutive direct inferences, first reduction and then conversion of the reduced judgement.

Predicate opposition is effected in the following way for the various types of judgement:

1. *A*. All *S* are *P*. → No *not-P* is not *S*.

All metals are electrically conductive. → A non-conductor is not a metal.

2. *E*. Any *S* is not *P*. → Some *not-P* are *S*.

Any death-cup is not edible. → Some inedible mushrooms are death-cups.

3. *O*. Some *S* are not *P*. → Some *not-P* are *S*.

Some crimes are not premeditated. → Some unpremeditated acts are crimes.

4. *I*. There are no necessary conclusions following from particular affirmative judgements.

Exercise

Obtain reduction, converse and predicate opposition for the following judgement: "All mushrooms are plants".

It is a judgement of type *A*.

Reduction: "No mushroom is not not-plant."

Converse (limited): "Some plants are mushrooms".

Predicate opposition: "Any non-plant is not a mushroom".

All types of direct inferences provide us with new knowledge, and this is especially true of predicate opposition.

§ 4. Categorical Syllogism

A *categorical syllogism* is a type of deductive inference in which a conclusion necessarily follows from the two true categorical judgements linked by the middle term provided that the rules are observed.

Syllogism comes from the Greek word meaning calculation or inference.

A categorical syllogism is made up of two premises and a conclusion.

All metals (*M*) are electrically conductive (*P*) – major premise.
Copper (*S*) is a metal (*M*) – minor premise.

Copper (*S*) is electrically conductive (*P*) – conclusion.

The concepts which make up a syllogism are called the syllogism's terms. In the example above, the terms are: *P* ("electrically conductive") – the major term, since it is the predicate of the conclusion; *S* ("copper") – the minor term, the subject of the conclusion; *M* ("metal") – the middle term, which serves to link *S* and *P* in the premises and is absent from the conclusion.

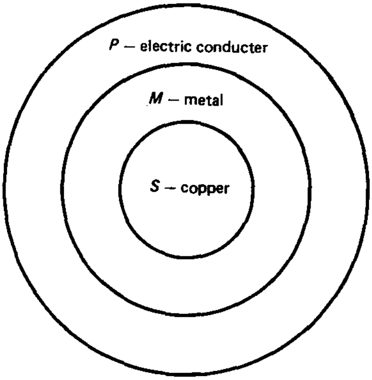

Fig. 50

The first premise, containing the predicate of the conclusion (i. e., the major term) is called the major premise. The second premise, containing the subject of the conclusion (i. e., the minor term) is called the minor premise.

The conclusion in a categorical syllogism is founded on the syllogistic axiom: "Everything which is asserted about a genus (or class) is necessarily affirmed or negated about a species (or a member of the given class) belonging to the said genus". In other words, what we assert about metal as a genus, we also affirm about its species – copper, in this case the property of being "electrically conductive".

Figures and modi of categorical syllogisms

The figures of the categorical syllogism denote its forms which differ according to the location of middle term *M* in the premises. There are four figures:

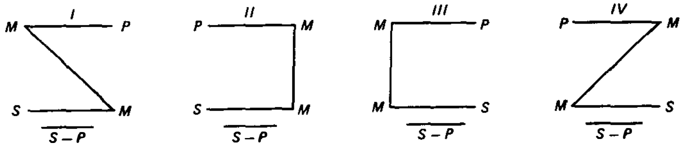

Fig. 51

Examples:

1. All cereals (*M*) are plants (*P*).
   Rye (*S*) is a cereal (*M*).
   ———
   Rye (*S*) is a plant (*P*).

2. All zebras (*P*) are striped (*M*).
   This animal (*S*) is not striped (*M*).
   ———
   This animal (*S*) is not a zebra (*M*).

3. All carbons (*M*) are simple bodies (*P*).
   All carbons (*M*) are electrical conductors (*S*).
   ———
   Some electrical conductors (*S*) are simple bodies (*P*).

4. All whales (*P*) are mammals (*M*).
   No mammals (*M*) are fish (*S*).
   ———
   No fish (*S*) are whales (*P*).

Specific rules of figures

Figure I.	The major premise must be universal and the minor premise affirmative.
Figure II.	The major premise is universal, and one of the premises as well as the conclusion are negative.
Figure III.	The minor premise must be affirmative and the conclusion particular.
Figure IV.	Does not produce any general affirmative conclusions. If the major premise is affirmative, then the minor premise must be universal. If one of the premises is negative, the major premise must be universal.

Modi of the categorical syllogism

Modi of the figures of the categorical syllogism is the name given to the varieties of syllogism which differ from each other according to the qualitative and quantitative nature of the premises and the conclusion.

In total, there are 19 valid modi for the four figures.

Figure I has the following valid modi (the letters denote the quantity and quality of the major premise, the minor premise and the conclusion respectively): *AAA, EAE, AII, EIO*. Example 1 illustrates modus *AAA*.

For Figure II, the valid modi are: *AEE, AOO, EAE* and *EIO*. Inference 2 is constructed according to modus *AEE*.

Figure III has the following valid modi: *AAI, EAO, IAI, OAO, AII* and *EIO*. Modus *AAI* is represented by example 3.

Figure IV has the valid modi *AAI, AEE, IAI, EAO* and *EIO*. Example 4 represents modus AEE.

Rules of the categorical syllogism

Categorical syllogisms are encountered extremely often in thought. In order to obtain a true conclusion, it is essential to take true premises and observe the rules of the categorical syllogism listed below (and also the specific rules of the figures as listed above).

I. Rules of terms

1. In each syllogism there must be only three terms (*S, P, M*). An error here is known as the "quadruplication of terms". The following is an erroneous inference:

Motion is eternal.
Walking to the institute is motion.

Walking to the institute is eternal.

Here, "motion" is interpreted in two different senses, the philosophical and the everyday one.

2. The middle term should be distributed at least with respect to one of the premises.

Some plants (*M*) are poisonous (*P*).
White mushrooms (*S*) are plants (*M*).

White mushrooms (*S*) are poisonous (*P*).

Here, the middle term "plant" is not distributed with respect to either of the premises, and so the conclusion is false.

3. A term which is not distributed in the premise cannot be distributed in the conclusion. Otherwise the terms of the conclusion would say more than the terms of the premises.

The midnight sun occurs in all towns north of the Arctic Circle.
Leningrad is not north of the Arctic Circle.

The midnight sun does not occur in Leningrad.

The conclusion is false because the said rule has been violated. The predicate in the conclusion is distributed, but in the premise it is not, so that the major term has been extended.

II. Rules of premises

4. It is not possible to draw any conclusion from two negative premises.

For example,

Dolphins are not fish.
Pike are not dolphins.

?

5. If one of the premises is negative, then the conclusion should also be negative.

All walruses are fin-footed.
This animal is not fin-footed.

This animal is not a walrus.

6. It is not possible to draw a conclusion from two particular premises.

Some animals are reptiles.
Some living organisms are animals.

?

7. If one of the premises is particular, the conclusion must be particular.

All lizards are reptiles.
Some animals are lizards.

Some animals are reptiles.

The most widespread mistakes in categorical syllogisms are the following:

1. The conclusion is drawn after Figure I from the minor negative premise.

All classrooms need ventilation.
This room is not a classroom.

This room does not need ventilation.

All students sit for exams.
Smirnov is not a student.

Smirnov does not sit for exams.

The conclusion does not necessarily follow from the premises, since the second premise is not affirmative.

2. The conclusion is drawn after Figure II with two affirmative premises:

All zebras are striped.
This animal is striped.

This animal is a zebra.

The conclusion does not necessarily follow from these premises, since one of the premises and the conclusion must be negative judgements.

§ 5. Incomplete Categorical Syllogisms (Enthymemes)

Enthymeme, or an incomplete categorical syllogism, is a syllogism in which one of the premises or the conclusion is missing.

The word enthymeme derives from the Greek and means "in mind", "in thoughts". The following inference is an example of an enthymeme: "Vasilyev is a citizen of the USSR: therefore he has the right to housing".

Let us restore the enthymeme:

All citizens of the USSR have the right to housing.
Vasilyev is a citizen of the USSR.

Vasilyev has the right to housing.

The major premise has been missing.

"The rights of Soviet inventors are protected by the state, and thus the rights of this man are protected by the state".

When restored, the enthymeme looks like this:

The rights of Soviet inventors are protected by the state.
The rights of this man are the rights of a Soviet inventor.

The rights of this man are protected by the state.

The minor premise has been missing.

The enthymeme "All fish breathe through their gills, and the perch is a fish" lacks the conclusion.

In restoring an enthymeme, we must first establish which judgement is the premise and which is the conclusion.

The premise usually follows conjunctions like "since", "because" and "as", and the conclusion follows "therefore", "for this reason", "that is why", etc.

Students are given an enthymeme: "This physical process is not vaporisation, since there is no transition of any substance from a liquid to steam". They restore the enthymeme, that is, formulate the complete categorical syllogism. The judgement following the word "since" is the premise. The missing part of the enthymeme is the major premise, which the students formulate on the basis of their knowledge of physical processes.

Vaporisation is the process by which a substance changes from a liquid to steam.
This physical process is not the process of a liquid changing to steam.

This physical process is not vaporisation.

This particular categorical syllogism is constructed after Figure II. Its specific rules have been observed, since one of the premises and the conclusion are negative, the major premise is universal and represents a definition of the concept "vaporisation".

Enthymemes are used more frequently than complete categorical syllogisms.

§ 6. Complex and Complex Incomplete Syllogisms (Polysyllogisms, Sorites and Epihairems)

Polysyllogism (complex syllogism) is the name given to two or more simple categorical syllogisms linked in such a way that the conclusion of one becomes the premise of the next. A distinction is made between progressive and regressive polysyllogisms.

In a *progressive polysyllogism*, the conclusion from the previous syllogism becomes the major premise of the subsequent one. Let us give an example of a progressive polysyllogism which is a chain of three syllogisms and may be represented schematically as follows:

	Pattern
Everything that strengthens the health (*A*) is beneficial (B).	All *A* are *B*.
Sport (*C*) strengthens the health (*A*).	All *C* are *A*.
Which means that sport (*C*) is beneficial (*B*).	Which means that all *C* are *B*.
Athletics (*D*) is sport (*C*).	All *D* are *C*.
Which means that athletics (*D*) is beneficial (*B*).	{ All *D* are *B*.
Running (*E*) is a kind of athletics (*D*).	{ All *E* are *D*.
Running (*E*) is beneficial (*B*).	All *E* are *B*.

Let us take a polysyllogism consisting of two syllogisms and write it in schematic form on the right.

	Pattern
All metals (*A*) are heat conductors (*B*).	All *A* are *B*.
Alkali-earth metals (*C*) are metals (*A*).	All *C* are *A*.
Alkali-earth metals (*C*) are heat conductors (*B*).	{ All *C* are *B*.
Calcium (*D*) is an alkali-earth metal (*C*).	{ All *D* are *C*.
Calcium (*D*) is a heat conductor (*B*)	All *D* are *B*.

If we replace general categorical judgements with conditional ones having the same meaning, the second polysyllogism will take on the following appearance:

If the object is a metal, it is a heat conductor.
If the object is an alkali-earth metal, it is a metal.
If the object is an alkali-earth metal, it is a heat conductor.
If the object is calcium, it is an alkali-earth metal.

If the object is calcium, it is a heat conductor.

If we denote the judgement "The object is a metal" with the letter *a*, the judgement "The object is a heat conductor" with *b*, the judgement "The object is an alkali-earth metal" with *c* and the judgement "The object is calcium" with *d*, we may obtain a formula from the algebra of logic which accords with the progressive polysyllogism above: $((a \rightarrow b) \wedge (c \rightarrow a) \wedge (c \rightarrow b) \wedge (d \rightarrow c)) \rightarrow (d \rightarrow b)$.

This formula is identically true if all premises of the polysyllogism are universal judgements.

A *regressive polysyllogism* is a complex syllogism in which the conclusion of the previous syllogism becomes the minor premise of the subsequent syllogism.

1.	2.
All organisms (*B*) are bodies (*C*). All plants (*A*) are organisms (*B*).	All bodies (*C*) possess weight (*D*). All plants (*A*) are bodies (*C*).
All plants (*A*) are bodies (*C*).	All plants (*A*) possess weight (*D*).

Let us represent these two syllogisms schematically:

1.	2.
All *B* are *C*. All *A* are *B*.	All *C* are *D*. All *A* are *C*.
All *A* are *C*.	All *A* are *D*.

By combining the two and avoiding repetition of the judgement "All *A* are *C*", we obtain the pattern of a regressive polysyllogism for universal affirmative premises:

All *B* are *C*;
All *A* are *B*.
{ All *C* are *D*.
{ All *A* are *C*.
———————
All *A* are *D*.

The formula is: $((b \rightarrow c) \wedge (a \rightarrow b) \wedge (c \rightarrow d) \wedge (a \rightarrow c)) \rightarrow (a \rightarrow d)$.

Sorites (with universal premises)

In thought, progressive and regressive polysyllogisms are more often than not used in a limited form, as sorites.

There exist two types of sorites: progressive and regressive.

A *progressive sorites* is obtained from a progressive polysyllogism by discarding the conclusions of the previous syllogisms and the major premises of the subsequent ones.

Everything which strengthens the health (A) is beneficial (B).
Sport (C) strengthens the health (A).
Athletics (D) is a sport (C).
Running (E) is a kind of athletics (D).

Running (E) is beneficial (B).

The pattern for a progressive sorites looks like this:

All A are B.
All C are A.
All D are C.
All E are D.

All E are B.

A progressive sorites begins with a premise containing the predicate of the conclusion and ends with a premise containing the subject of the conclusion. Its formula is:

$$((a \rightarrow b) \wedge (c \rightarrow a) \wedge (d \rightarrow c) \wedge (e \rightarrow d)) \rightarrow (e \rightarrow b).$$

A *regressive sorites* is obtained from a regressive polysyllogism by discarding the conclusions of the previous syllogisms and the minor premises of the subsequent ones. In the first categorical syllogism, the premises change places.

All plants (A) are organisms (B).
All organisms (B) are bodies (C).
All bodies (C) possess weight (D).

All plants (A) possess weight (D).

The pattern of a regressive sorites is:

All A are B.
All B are C.
All C are D.

All A are D.

A regressive sorites starts with a premise containing the subject of the conclusion and ends with a premise containing the predicate of the conclusion.

$$((a \rightarrow b) \wedge (b \rightarrow c) \wedge (c \rightarrow d)) \rightarrow (a \rightarrow d).$$

This formula of the algebra of logic (or propositional calculus) accords with a regressive sorites consisting of three universal affirmative premises.

The formalisation of epihairems with universal premises

An *epihairem* in traditional logic is the name given to an incomplete complex syllogism both premises of which represent incomplete simple syllogisms (enthymemes).

The pattern of an epihairem containing just universal and affirmative propositions is usually written like this:

All *A* are *C*, since *A* is *B*.
All *D* are *A*, since *D* is *E*.

All *D* are *C*.

An example of an epihairem:

Noble work (*A*) deserves respect (*C*), since noble work (*A*) furthers the advance of society (*B*).

The work of a teacher (D) is noble work (*A*) since the work of a teacher (*D*) is part and parcel of the education and upbringing of the younger generation (*E*).

The work of a teacher (*D*) deserves respect (*C*).

The first and second premises of an epihairem are enthymemes, i. e., incomplete categorical syllogisms with one of the premises omitted. Let us express the first and second enthymemes of an epihairem in full.

1.	2.
All *B* are *C*. All *A* are *B*.	All *E* are *A*. All *D* are *E*.
All *A* are *C*.	All *D* are *A*.

Let us take the conclusions of the first and second syllogisms and make them into the major and minor premises of a new, third syllogism:

3. All *A* are *C*.
All *D* are *A*.

All *D* are *C*.

Let us restore the epihairem in full:

1. Everything which furthers the advance of society (*B*) deserves respect (*C*).
Noble work (*A*) furthers the advance of society (*B*).

Noble work (*A*) deserves respect (*C*).

2. The teaching and upbringing of the younger generation (*E*) is noble work (*A*).
The work of a teacher (*D*) is part and parcel of the teaching and upbringing of the younger generation (*E*).

The work of a teacher (*D*) is noble work (*A*).

The conclusions of the first and second syllogisms form the premises of the third syllogism.

3. Noble work (*A*) deserves respect (*C*).
The work of a teacher (D) is noble work (A).

The work of a teacher (*D*) deserves respect (*C*).

Let us give another example of an epihairem.

All fish (*A*) are vertebrates (*C*), since fish (*A*) have a skeleton (*B*).
All sharks (*D*) are fish (*A*), since sharks (*D*) breathe through their gills (*E*).

All sharks (*D*) are vertebrates (*C*).

In conformity with the rules of inference, the restored epihairem can be written like this:

$$\frac{\begin{array}{l} b \to c,\ a \to b \vdash a \to c \\ e \to a,\ d \to e \vdash d \to a \end{array}}{d \to c}$$

(Here "$\vdash$" is the inference sign.) The following formula accords with the rule:

$$((b \to c) \land (a \to b) \land (e \to a) \land (d \to e)) \to (d \to c).$$

For the sake of greater clarity, let us transposit the premises and write the formula as follows:

$$((d \to e) \land (e \to a) \land (a \to b) \land (b \to c)) \to (d \to c).$$

It can be proven that this formula is a law of logic.

Just like enthymemes, complex incomplete syllogisms make our arguments much easier to formulate.

Inferences based on logical relations between judgements (inferences of propositional logic).

Whilst in predicational logic, simple judgements are split up into subject and predicate, judgements (propositions) are not divisible in propositional logic, but are viewed as simple judgements which can be made to form complex judgements by means of logical connectives (logical constants).

The rules of direct inference in propositional logic allow a true conclusion to be drawn from the true premises. On the basis of the rules of direct inference it is possible to construct purely conditional, conditional-categorical, disjunctive and disjunctive-categorical as well as conditional-disjunctive (lemmatic) inferences.

§ 7. Conditional Inferences

Purely conditional (hypothetical) inference is mediated inference in which both premises and the conclusion are conditional judgements. A proposition is considered conditional if it has the structure "If *a*, then *b*". The structure of purely conditional inference is as follows:

	Pattern
If *a*, then *b*.	
If *b*, then *c*.	$a \rightarrow b$, $b \rightarrow c$
If *a*, then *c*.	$a \rightarrow c$.

This type of inference is often used in teaching, notably in the study of mathematics, physics and other sciences. Let us give an example:

If electric current is passed through a conductor, a magnetic field will be produced around the conductor.
If a magnetic field is produced around the conductor, iron filings will align themselves in this magnetic field along the lines of force.

If electric current is passed through a conductor, iron filings will align themselves in its magnetic field along the lines of force.

A purely conditional inference has various classes (modi). These include, for example, the following:

	Pattern
If *a*, then *b*.	$a \to b$
If *not-a*, then *b*.	$\bar{a} \to b$
b	*b*

The formula is $((a \to b) \wedge (\bar{a} \to b)) \to b$.

This formula is a law of logic. In this inference, judgement *b* is true independently of whether *a* is affirmed or negated.

The following argument is an example of such inference:

If the weather is good, we will gather in the harvest.
If the weather is not good, we will gather in the harvest.

We will gather in the harvest.

Given four premises, the formula of purely conditional inference will be written as follows:

$$((a \to \mathrm{b}) \wedge (b \to c) \wedge (c \to d) \wedge (d \to e)) \to (a \to e).$$

Conditional-categorical inference

Conditional-categorical inference is deductive inference in which one premise is a conditional judgement and the other is a simple categorical judgement.

It has two valid modi giving a conclusion necessarily following from the premises.

I. Positive modus (*modus ponens*).

Its structure is: If *a*, then *b*. Pattern: $a \to b$.

$$\frac{\mathrm{a}}{\mathrm{b}}$$

Formula (1): $((a \to b) \wedge a) \to b$ is a law of logic.

A necessary inference can be made from the assertion of the antecedent to the assertion of the consequent.

For example:

If this metal is silver (*a*), it activates oxygen so as to destroy bacteria (*b*).

This metal is silver (*a*).

This metal activates oxygen so as to destroy bacteria (*b*).

Any use of rules in Russian, English, French and all other languages, as well as in mathematics, physics, chemistry and other sciences, is based on the positive modus, which produces a necessary conclusion, and for this reason is extremely widespread in thought.

If this metal is sodium, it is lighter than water.
This metal is sodium.

This metal is lighter than water.

II. Negative modus (*modus tollens*).

Its structure is: If a, then b Pattern $a \to b$

Structure	Pattern
If a, then b	$a \to b$
Not-b	$\bar{b}$
Not-a	$\bar{a}$

Formula (2): $((a \to b) \wedge \bar{b}) \to \bar{a}$ is also a law of logic (this can be shown by means of a table).

A necessary inference can be made from the negation of the consequent to the negation of the antecedent.

For example:

If this figure is a rhombus (a), its diagonals are perpendicular to each other (b).

In this figure, the diagonals are not perpendicular to each other ($\bar{b}$).

This figure is not a rhombus ($\bar{a}$).

A conditional-categorical inference may not only produce a valid conclusion, but also a probable one.

First probable modus

Structure	Pattern
If a, then b	$a \to b$
b	b
Probably a	Probably a

Formula (3): $((a \to b) \wedge b) \to a$ is not a law of logic.

A necessary inference cannot be made from the assertion of the consequent to the assertion of the antecedent. For example, in the inference,

If the bay has frozen up, ships cannot enter the bay.
Ships cannot enter the bay.

It is probable that the bay has frozen up, the conclusion will just be a probable judgement, i. e., it is probable that the bay is frozen up, but it is possible that a gale is blowing or the bay has been mined, or there may be some other reason why ships cannot enter the bay.

The following inference provides a probable conclusion.

If this body is graphite, it is an electrical conductor.
This body is an electrical conductor.

It is probable that this body is graphite.

Second probable modus

Structure:	If *a*, then *b*	Pattern: $a \rightarrow b$
	Not-a	$\bar{a}$
	Probably not-b	Probably $\bar{b}$

Formula (4): $((a \rightarrow b) \wedge \bar{a}) \rightarrow \bar{b}$ is not a law of logic.

A necessary inference cannot be made from the negation of the antecedent to the negation of the consequent. For example,

If a man has a high temperature, he is ill.
This man does not have a high temperature.

This man is probably not ill.

People sometimes make logical errors in making an inference. They will infer as follows:

If a body is subjected to friction, it becomes warm.
This body has not been subjected to friction.

This body has not become warm.

The conclusion is just probable and not necessary, since a body may become warm for some other reason (from the sun, in an oven, etc.).

Let us note that such examples are quite sufficient to show that the forms of inferences expressed by formulas (3) and (4) are invalid. But no amount of examples after formulas (1) and (2)–if we rely only on examples–can substantiate their logical validity. This substantiation requires some logical theory. Whilst virtually non-existent in traditional logic, such theory is to be found in the algebra of logic. If a formula in which the conjunction of the premises and the supposed conclusion are linked by an implication sign is not identically true, i.e., does not express a law of logic, the conclusion in the inference is not necessary. In the table of verity, it can be seen that the columns corresponding to formulas (1) (*modus ponens*) and (2) (*modus tollens*) consist entirely of "T" ("true") signs, so that (1) and (2) express laws of logic, which means that *modus ponens* and *modus tollens* are valid forms of inference.

a	b	$\bar{a}$	$\bar{b}$	$a \to b$	$(a \to b) \wedge \bar{a}$	$((a \to b) \wedge \bar{a}) \to \bar{b}$	$(a \to b) \wedge \bar{b}$	$((a \to b) \wedge \bar{b}) \to \bar{a}$
T	T	F	F	T	T	T	F	T
T	F	F	T	T	F	T	F	T
F	T	T	F	T	F	T	F	T
F	F	T	T	T	F	T	T	T

The table for the invalid modi is as follows

a	b	$\bar{a}$	$\bar{b}$	$a \to b$	$(a \to b) \wedge \bar{a}$	$((a \to b) \wedge \bar{a}) \to \bar{b}$	$(a \to b) \wedge b$	$((a \to b) \wedge b) \to a$
T	T	F	F	T	F	T	T	T
T	F	F	T	F	F	T	F	T
F	T	T	F	T	T	F	T	F
F	F	T	T	T	T	T	F	T

Together with the sign "T", we also find the sign "F" ("false"), which means that the propositions $((a \to b) \wedge b) \to a$ and $((a \to b) \wedge \bar{a}) \to \bar{b}$ are not identically true statements and therefore do not represent laws of logic.

If an inference is made from the affirmation of the consequent to the affirmation of the antecedent, a false conclusion may be obtained due to the multiplicity of causes which may produce one and the same effect. For example, to establish the cause why someone has become ill, it is necessary to consider all possible causes: he may have caught a cold, overworked, been in contact with a bacillus carrier, etc.

§ 8. Disjunctive Inferences

Disjunctive is the name given to an inference in which one or more premises are disjunctive judgements. There exist purely disjunctive and disjunctive-categorical inferences.

In a *purely disjunctive* inference, both (or all) premises are disjunctive judgements.

In traditional logic, such an inference is accepted to have the following structure:

S is A, or B, or C.
A is either A_1, or A_2.

S is either A_1, or A_2, or B, or C.

In the first disjunctive judgement, each of the three simple judgements; S is A, S is B, S is C, is called an alternative. Two more alternatives are formed from judgement "S is A", which become the two terms of the new disjunction.

If we denote judgement "S is A" by the letter a, judgement "S is B" by b, "S is C" by c, "A is A_1" by a_1 and "A is A_2" by a_2, we obtain the pattern of a purely disjunctive inference:

$a \vee b \vee c$
a_1 a_2

$a_1 \vee a_2 \vee b \vee c$

Example:

Any philosophical system is either idealism or materialism.
The idealist system is either objective or subjective idealism.

Any philosophical system is either objective idealism, or subjective idealism or materialism.

In a *disjunctive-categorical* inference, one of the premises is a disjunctive judgement and the other is a simple categorical judgement. This type of inference has two modi.

Modus I – positive-negative (*modus ponendo tollens*).

This verb may occur in the present, past or future tense.
This verb is in the present tense.

This verb is neither in the future nor in the past tense.

By replacing the concrete propositions in the premises and the conclusion with variables, we obtain a notation of this modus (with two members of disjunction), in the categories of symbolic logic, having the form of a rule of inference:

$$\frac{a \dot{\vee} b,\ a}{\bar{b}} \quad \text{or} \quad \frac{a \dot{\vee} b,\ b}{\bar{a}}.$$

In this modus, the conjunction "or" is used in the sense of exclusive disjunction. The formulas corresponding to this modus look like this:

(1) $((a \dot{\vee} b) \wedge a) \rightarrow \bar{b}$; (2) $((a \dot{\vee} b) \wedge b) \rightarrow \bar{a}$.

Both these formulas express laws of logic.

If the conjunction "or" is absent from this when taken to mean inclusive disjunction, then formulas (3) and (4), which accord with this will not express a law of logic.

(3) $((a \vee b) \wedge a) \rightarrow \bar{b}$; (4) $((a \vee b) \wedge b) \rightarrow \bar{a}$.

The proof for formulas (1) and (3) is given in the following table:

a	b	$\bar{b}$	$a \vee b$	$(a \vee b) \wedge a$	$((a \vee b) \wedge a) \rightarrow \bar{b}$	$a \dot{\vee} b$	$(a \dot{\vee} b) \wedge a$	$((a \dot{\vee} b) \wedge a) \rightarrow \bar{b}$
T	T	F	T	T	F	F	F	T
T	F	T	T	T	T	T	T	T
F	T	F	T	F	T	T	F	T
F	F	T	F	F	T	F	F	T

Mistakes occur due to the confusion in this modus of the conjunctive-disjunctive and the exclusive-dusjunctive meaning of the conjunction "or". It is inadmissible, for example, to argue in the following way:

> In mathematics tests students make mistakes either in calculation, or in equivalent transformations or in the application of the algebraic rules they have learned.
>
> Student Sidorov made mistakes in calculation in his mathematics test.
>
> ---
>
> In his test, Sidorov made no mistakes in equivalent transformations or in the application of the algebraic rules he had learned.

The conclusion is not valid, since Sidorov may have made all three types of mistakes.

Modus II is negative-positive (*modus tollendo ponens*). Here is an example:

Fertilisers may be nitrogenous, phosphatic or potassic.
This fertiliser is neither nitrogenous nor phosphatic.

This fertiliser is potassic.

The negative-affirmative modus (for the two-member disjunctive premise) may be written as follows in the form of a rule of inference in the algebra of logic:

$$\frac{a \vee b, \bar{a}}{b}; \quad \frac{a \vee b, \bar{b}}{a}; \quad \frac{a \dot{\vee} b, \bar{a}}{b}; \quad \frac{a \dot{\vee} b, \bar{b}}{a}.$$

Here, the logical conjunction "or" may be used in two senses: as exclusive disjunction ($\dot{\vee}$) or an inclusive disjunction ($\vee$), i. e., the nature of disjunction does not affect the necessity of the conclusion in this modus.

There are four formulas corresponding to this modus, all of which are laws of logic:

(1) $((a \vee b) \wedge \bar{a}) \rightarrow b;$ (3) $((a \dot{\vee} b) \wedge \bar{a}) \rightarrow b;$

(2) $((a \vee b) \wedge \bar{b}) \rightarrow a;$ (4) $((a \dot{\vee} b) \wedge \bar{b}) \rightarrow a.$

Following a disjunctive-categorical inference, it is an essential condition to abide by the rule that all possible alternatives must be envisaged in the disjunctive premise, i. e., the division must be complete. This rule is binding for the negative-positive modus.

Example:

The fire may have happened due to negligence with a naked flame or as a result of arson or because of a faulty electric wiring.
The fire in question did not happen either as a result of negligence with a naked flame or because of a faulty electric wiring.

The fire happened as a result of arson.

The conclusion is not necessary but probable, since the first disjunctive premise does not list all the causes for a fire (for example, due to an explosion or lightning, etc.).

The following are examples of problems being solved by means of conditional-categorical and disjunctive-categorical inferences.

Problem 1 (see p. 216).

If the animal is a mammal (a), it belongs to the chordata family (b).
This animal is not a mammal ($\bar{a}$).

This animal does not belong to the chordata family ($\bar{b}$).

The simple judgements and their negations are denoted by small letters. This problem can be solved in two ways.

The first way is to follow the rules of traditional logic. It is a conditional-categorical inference. The modus is probable, since one cannot necessarily infer a negative consequent from a negative antecedent.

The second way is to use symbolic logic. Let us construct the pattern and formula of this inference.

Pattern: Formula:

$a \rightarrow b, \bar{a}$ $((a \rightarrow b) \wedge \bar{a}) \rightarrow \bar{b}$.

Probably $\bar{b}$

We now need to prove whether this formula is a law of logic.

a	b	$\bar{a}$	$\bar{b}$	$a \rightarrow b$	$(a \rightarrow b) \wedge \bar{a}$	$((a \rightarrow b) \wedge \bar{a}) \rightarrow \bar{b}$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	T	F
F	F	T	T	T	T	T

It follows: Since in the last column we find one instance of "false", this formula is not a law of logic. This means that the inference is invalid and the conclusion is not a necessary proposition but only a probable one.

Problem 2.

Depending on their scale, maps are divided into large-scale, medium-scale and small-scale.

This map is neither large-scale nor medium-scale.

This map is small-scale.

The type of the inference is disjunctive-categorical, and the modus is negative-affirmative.

Pattern: Formula:

$$\frac{a \dot{\vee} b \dot{\vee} c, \bar{a} \wedge \bar{b}}{c} \qquad ((a \dot{\vee} b \dot{\vee} c) \wedge (\bar{a} \wedge \bar{b})) \rightarrow c.$$

Proof of the formula:

a	b	c	$\bar{a}$	$\bar{b}$	$a \dot{\vee} b \dot{\vee} c$	$\bar{a} \wedge \bar{b}$	$(a \dot{\vee} b \dot{\vee} c) \wedge (\bar{a} \wedge \bar{b})$	Entire formula
T	T	T	F	F	F	F	F	T
T	T	F	F	F	F	F	F	T
T	F	T	F	T	F	F	F	T
T	F	F	F	T	T	F	F	T
F	T	T	T	F	F	F	F	T
F	T	F	T	F	T	F	F	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	F	T	F	T

Since we only find the value "true" in the final column, the said formula is a law of logic. Consequently, the disjunctive-categorical inference is correctly constructed and the conclusion is a true judgement. *Explanation*: Exclusive disjunction consisting of three elements ($a \dot{\vee} b \dot{\vee} c$) is true when one and only one of the three judgements is true. This accords with the meaning of the connective "or" in exclusive disjunctive judgements expressed in natural language, which for a three-member disjunction can be expressed by the following formula:

$$(a \dot{\vee} b \dot{\vee} c) \equiv (a \vee b \vee c) \wedge \overline{a \wedge b} \wedge \overline{a \wedge c} \wedge \overline{b \wedge c} \, (1).$$

The word "or" in natural language expresses the fact that one and only one of the three judgements can obtain in practice. The two others are false or, in other words, the three judgements are incompatible taken in pairs, but one of them is true. A similar meaning accords to the exclusive-disjunctive substantive connective "or" in the case of disjunctive judgements involving four or five, etc. members. For a general case: of n judgements linked by the exclusive-disjunctive connective "or" in natural language, one and only one is true. This may be seen from a table compiled for a three-member disjunction with an exclusive "or" (see formula (1) above).

a	b	c	$a \vee b \vee c$	$\overline{a \wedge b}$	$\overline{a \wedge c}$	$\overline{b \wedge c}$	$(a \vee b \vee c) \wedge \overline{a \wedge b} \wedge \wedge \overline{a \wedge c} \wedge \overline{b \wedge c}$	$a \dot{\vee} b \dot{\vee} c$	Entire formula
T	T	T	T	F	F	F	F	F	T
T	T	F	T	F	T	T	F	F	T
T	F	T	T	T	F	T	F	F	T
T	F	F	T	T	T	T	T	T	T
F	T	T	T	T	T	F	F	F	T
F	T	F	T	T	T	T	T	T	T
F	F	T	T	T	T	T	T	T	T
F	F	F	F	T	T	T	F	F	T

But in the algebra of logic, there are divergences from the meaning of the exclusive "or" described above in exclusive-disjunctive judgements of natural language. Formulas $a \dot{\vee} b \dot{\vee} c$, $a \dot{\vee} b \dot{\vee} c \dot{\vee} d$, etc., *in the algebra of logic*, are true when and only when an odd number of disjunctive members takes the value "true". Given the law of associativity, formula $a \dot{\vee} b \dot{\vee} c$ for exclusive disjunction can be written as $(a \dot{\vee} b) \dot{\vee} c$. When examined in this light, $a \dot{\vee} b$ will be false if both a and b are true, and in exclusive disjunction the falsity of $(a \dot{\vee} b)$ and the truth of c give us a true judgement. Thus, if we have three true members, disjunction $a \dot{\vee} b \dot{\vee} c$ will be true in the algebra of logic, since three is an odd number.

Aware of this divergence, in the case of disjunctive judgements we shall use the rule for natural language and regard exclusive disjunction (for any number of its members) as being true when one and only one judgement is true. This accords with the sense in which exclusive disjunction is used in disjunctive-categorical judgements.

§ 9. Conditional-Disjunctive (Lemmatic) Inferences

Conditional-disjunctive inference is one in which one of the premises is made up of two or more hypothetical judgements and the other is a *disjunctive judgement*. Depending on the number of members in the disjunctive premise, this judgement may be a dilemma (if the disjunctive premise contains two members), a trilemma (if the disjunctive premise contains three members) or a polylemma (the name used in all cases when there are more than two disjunctive members).

The formalisation of a dilemma

Dilemmas may be constructive and destructive; both forms may in turn be simple or complex.

Simple constructive dilemma

This inference consists of two premises. In the first premise it is affirmed that one and the same consequent depends on two different antecedents. The second premise, which is a disjunctive judgement, asserts that one or the other of these antecedents is true. The conclusion asserts the consequent.

In traditional formal logic, a simple constructive dilemma is represented as follows:

If *A* is *B*, then *C* is *D*; if *E* is *F*, then *C* is *D*.
A is *B* or *E* is *F*.

C is *D*.

Let us give an example of a simple constructive dilemma.

Doctor's reasoning after examining a patient.
If the patient has strained ligaments, he is recommended to undergo paraffin treatment; if the patient has a contusion, he is recommended to undergo paraffin treatment.
This patient has strained ligaments or a contusion.

This patient is recommended to undergo paraffin treatment.

Let us express judgement "*A* is *B*" with variable *a*, "*C* is *D*" with variable *b* and "*E* is *F*" with variable *c*. The pattern of a simple constructive dilemma will then take the form of the following rule of inference:

$$\frac{a \rightarrow b,\ c \rightarrow b,\ a \vee c}{b}.$$

In conformity with the definition of logical consequent formulated in propositional calculus (the algebra of logic), if *b* is the logical consequent of the given premises, then by joining up the premises with a conjunction sign and attaching the consequent to them with an implication sign, we should obtain a formula which expresses a law of logic, i. e., an identically true formula. In this case, the formula of the type indicated will be:

$$((a \rightarrow b) \wedge (c \rightarrow b) \wedge (a \vee c)) \rightarrow b.$$

The identical truth of this formula can be proven by compiling a table.

Let us give another example of a simple constructive dilemma:

If I cross the river by the bridge, I may be noticed; if I wade across the river, I may also be noticed.
I may cross the river by the bridge or wade across it.

I may be noticed.

Complex constructive dilemma

This is an inference made up of two premises. The first premise has two antecedents from which there accordingly derive two consequents; the second premise is a disjunctive judgement and asserts the truth of one or the other antecedent; the conclusion asserts the truth of one or the other consequent. A complex constructive dilemma differs from a simple one merely to the extent that the two consequents in its conditional premise are different and not identical.

An example of a complex constructive dilemma is the following argument:

If I finish the drawings tonight, I shall be tired at work next morning; but if I rest tonight, I shall not complete the work on time.
Tonight I may finish the drawings or rest.

I shall either be tired at work next morning or not complete the work on time.

Since a dilemma is a complex choice of one of two alternatives, both of which are undesirable for the subject (this situation may be described by the saying "choose the lesser of two evils"), in ancient times people used to speak about "sitting on the horns of a dilemma". In more recent times, we say, "I am facing a dilemma", (i. e., a difficult choice).

The pattern of a complex constructive dilemma is:

$$\frac{a \to b,\ c \to d,\ a \vee c,}{b \vee d}$$

Formula: $((a \to b) \wedge (c \to d) \wedge (a \vee c)) \to (b \vee d)$.

This formula is a law of logic, as may be proven using a table.

Let us give examples of the dilemmas which faced the famous physicist and chemist Marie Curie. Together with her husband Pierre, she discovered polonium and radium and studied radioactive radiation. In early August 1914, when the Germans invaded France without declaring war, Marie Curie was staying in Paris and her daughters Irène and Eve were at the summer house in Brittany. Quickly familiarising herself with the organisation of medical care, she found something lacking which seemed quite tragic to her: the field hospitals of the front line were completely without X-ray equipment. The way to bridge the gap was to apply the scientific achievements recorded by her husband and herself. Marie understood the enormous importance of maintaining her laboratory.

In the biography *Madame Curie*, written by her daughter Eve, this situation and the dilemmas facing the physicist are described as follows: "The rapid advance of the Germans gave Marie's conscience a problem to decide. Should she stay in Paris or go to join her daughters in Brittany? And if the enemy threatened to occupy the capital, should she follow the retreat of the medical organisations?

"She calmly considered these alternatives and took her decision: she would remain in Paris whatever happened. It was not only the benevolent task she had undertaken that kept her; she was thinking of her laboratory, of her delicate instruments in the Rue Cuvier and of the new halls of the Rue Pierre Curie. 'If I am there,' she thought, 'perhaps the Germans will not dare plunder them: but if I go away, everything will disappear.' ... To be afraid was to serve the adversary. Nothing in the world would induce her to give a triumphant enemy the satisfaction of occupying a deserted Curie laboratory.

"She confided her daughters to her brother-in-law

Jacques, preparing them for a possible separation."[1] It is well known that no other female scientist enjoyed the same popularity as Marie Curie. She was awarded 10 prizes and 16 medals, as well as being elected an honorary member of 106 scientific institutions, academies and scientific societies.

Let us formulate the dilemmas which faced Marie Curie at the time in question:

First dilemma

If I stay in Paris, I shall have to part from my daughters, and if I join my daughters in Brittany, then the laboratory in Paris (the life's work of my late husband and myself) may be plundered by the invaders.

I may stay in Paris or join my daughters in Brittany.

I will either have to part from my daughters or the laboratory in Paris may be plundered by the invaders.

Second dilemma

If I leave Paris together with the withdrawing medical organisations, the laboratory in Paris will remain unattended, and if I remain at the laboratory in Paris I shall not be able to help the wounded soldiers in hospitals by using the X-ray equipment there.

I may leave Paris together with the withdrawing medical organisations or remain at the laboratory in Paris.

Either the laboratory in Paris will remain unattended or I shall be unable to help the wounded soldiers in hospitals by using the X-ray equipment there.

Marie Curie decided to stay in Paris whatever happened. This is how she bravely resolved the dilemma she was facing.

People face dilemmas, whether simple or complex, almost every day. Sometimes human life depends on their resolution. But the way in which someone solves a dilemma shows his moral standards, his attributes as a human being, his business and other qualities.

Simple destructive dilemma

In this inference, the first (conditional) premise indicates that one and the same antecedent gives rise to two

[1] *Madame Curie. A Biography by Eve Curie*, Garden City Publishing Co., Inc., Garden City, New York, 1943, pp. 291-292.

different consequents; the second premise is a disjunction of negations of both these consequents; the conclusion negates the antecedent.

Example:

If someone is suffering from typhus, then on the fourth to sixth day of the illness he will have a high temperature and a rash will appear.
The patient has no high temperature or no rash.

This person is not suffering from typhus.

The pattern of this dilemma is:

$$\frac{a \rightarrow (b \wedge c);\ \bar{b} \vee \bar{c}}{\bar{a}}.$$

This accords with the formula:

$$((a \rightarrow (b \wedge c) \wedge (\bar{b} \vee \bar{c}) \rightarrow \bar{a}$$

A simple destructive dilemma may also be constructed according to another pattern:

$$\frac{a \rightarrow b,\ a \rightarrow c,\ \bar{b} \vee \bar{c}}{\bar{a}}.$$

This accords with the formula:

$$((a \rightarrow b) \wedge (a \rightarrow c) \wedge (\bar{b} \vee \bar{c})) \rightarrow \bar{a}.$$

Example:

If a man has principles, then, seeing a friend's shortcomings, he will tell him about them and help him to overcome them.
Seeing a friend's shortcomings, the man either did not tell him about them or did not help him to overcome them.

This is not a man of principles.

Complex destructive dilemma

A dilemma of this kind contains one premise consisting of two conditional judgements with different antecedents and different consequents; the second premise is a disjunction of negations of the two consequents; the conclusion is a disjunction of negations of the two antecedents. In traditional logic, a complex destructive dilemma acquires the following form:

If A is B, then C is D; if E is F, then K is M.
C is not D or K is not M.

A is not B or E is not F.

The following inference is an example of an argument based on the complex destructive form of dilemma:

If Petrov is honest, then, having not completed the work today, he will admit this, and if Petrov is conscientious, then he will complete the work by next time.

But Petrov did not admit that he had not completed the work today and did not complete it by the next time.

Petrov is not honest or not conscientious.

The pattern of a complex destructive dilemma is as follows:

$$\frac{a \to b,\ c \to d,\ \bar{b} \vee \bar{d}}{\bar{a} \vee \bar{c}}.$$

This accords with the formula:

$$((a \to b) \wedge (c \to d) \wedge (\bar{b} \vee \bar{d})) \to (\bar{a} \vee \bar{c}),$$

which is a law of logic.

In the four patterns above, which correspond to the four types of dilemma, the conjunction "or" in the second (disjunctive) premise is taken to form an inclusive disjunction ($\vee$). Will the formulas expressed in the algebra of logic and according with the four types of dilemma be identically true if the conjunction "or" is used in the exclusive-disjunctive sense, i. e., if we take it to signify exclusive disjunction ($\dot{\vee}$)? Are the following formulas laws of logic:

(1) $(a \to b) \wedge (c \to b) \wedge (a \dot{\vee} c) \to b$,
(2) $(a \to b) \wedge (a \to c) \wedge (\bar{b} \dot{\vee} \bar{c}) \to \bar{a}$,
(3) $(a \to b) \wedge (c \to d) \wedge (a \dot{\vee} c) \to (b \dot{\vee} d)$,
(4) $(a \to b) \wedge (c \to d) \wedge (\bar{b} \dot{\vee} \bar{d}) \to (\bar{a} \dot{\vee} \bar{c})$?

(Since conjunction forms a "closer" connection than implication, the brackets may be omitted.)

The author of this book has shown that simple dilemmas (both constructive and destructive) represent laws of logic no matter what kind of disjunction (exclusive or inclusive) is found in the relevant formulas. Laws of logic are present in complex dilemmas (both constructive and destructive) only when the conjunction "or" is viewed as a case of inclusive disjunction. However, in the course of an argument constructed in the form of complex dilemma, people use exclusive disjunction, since they are faced with two mutually exclusive possibilities (both of which are undesirable). This discrepancy results from the fact that the connective "if... then" and the meaning of material implication do not entirely coincide (in two-valued logic).

Some logicians consider a dilemma to be the following type of inference:

If *A* is *B*, then *C* is *D*; if *E* is *F*, then *G* is *H*.
But *C* is not *D* and *G* is not *H*.

Consequently, *A* is not *B* and *E* is not *F*.

Example:

If I were rich, I would buy a car.
If I were dishonest, I would steal one.
But I have not bought one and not stolen one either.

I am not rich and not dishonest.

But in this case, the second premise and the conclusion are conjunctive, and not disjunctive judgements (which should be the case according to the rules for building a dilemma), because the above example is not a dilemma, since it contains no disjunctive premise which is characteristic of a dilemma. This inference is the simple sum of two conditional-categorical inferences constructed according to the *modus tollens* rule, and gives a true conclusion. The *modus tollens* formula is as follows:

$$((a \rightarrow b) \wedge \bar{b}) \rightarrow \bar{a}.$$

1. If I were rich, I would buy a car.
   I will not buy a car.

I am not rich.

2. If I were dishonest, I would steal a car.
   I will not steal a car.

I am not dishonest.

We are thus dealing with a conditional-conjunctive and not a conditional-disjunctive (lemmatic) inference.

Trilemma

Just like dilemmas, trilemmas may be constructive or destructive; each of these forms may in turn be simple or complex. A *simple constructive trilemma* consists of two premises and a conclusion. The first premise states that one and the same consequent derives from three different antecedents; the second premise is a disjunction of these three antecedents, and the conclusion asserts the consequent.

The following reasoning by a doctor will serve as an example of a trilemma.

> If the patient has the flu, he is recommended to take some medicine; if the patient has a respiratory illness, he is recommended to take some medicine; if the patient has quinsy, he is recommended to take some medicine.
>
> The patient has either the flu, or a respiratory illness or quinsy.
>
> ---
>
> The said patient is recommended to take some medicine.

In *a complex constructive trilemma*, the first premise consists of three different antecedents and three different consequents deriving from them. The second premise is a disjunctive judgement asserting (at least) one of the three antecedents. The conclusion affirms (at least) one of the three consequents.

The following is an example of a complex constructive trilemma:

In some stories, we find references to inscriptions at the junction of three roads which may, for example, contain this type of trilemma:

> He who goes straight on will suffer cold and hunger; he who turns right will remain unscathed but his horse will be killed; he who turns left will be killed but his horse will remain unscathed.
>
> He may go either straight on, turn right or left.
>
> ---
>
> He will either suffer hunger and cold, or he will remain unscathed but his horse will be killed, or he will be killed but his horse will remain unscathed.

Just like destructive dilemmas, destructive trilemmas may be simple or complex. Their structure is similar to the structure of a dilemma, but provides for three possible alternatives.

Let us give an example of a simple destructive trilemma.

If the weather deteriorates presently, his joints will hurt, his blood pressure will rise and there will be pain in the small of his back.

It is known that his joints either do not hurt, or his blood pressure has not risen, or there is no pain in the small of his back.

The weather will not deteriorate presently.

In mathematics, the structure of a trilemma is used when three possible ways emerge of resolving a problem or proving a theorem and it is necessary to select one of them.

Incomplete conditional, disjunctive and conditional-disjunctive inferences

In thought, a categorical syllogism is often used in an incomplete form, in the form of an enthymeme. It is possible to shorten not only categorical syllogisms, but also conditional, disjunctive and conditional-disjunctive inferences by omitting either one of the premises or the conclusion. Let us give examples of such incomplete inferences.

1. *The conclusion is omitted from the inference*

"If the said body is a metal, it expands on heating. The said body is a metal." The conclusion "The said body expands on heating" is not formulated explicitly, but is implicitly understood in this conditional-categorical inference.

The following disjunctive-categorical inference also has the conclusion missing: "Polygons may be regular or irregular. The said polygon is irregular." The conclusion "The said polygon is not regular" is omitted and may be readily restored.

In dilemmas and trilemmas, the conclusion may also not be explicitly formulated but implicitly understood. For example, in the complex destructive dilemma given below, the conclusion is not stated explicitly:

"If the rules of grain storage are observed, its self-ignition will not occur, and if the grain stores are well guarded, there will be no arson. The fire in question occurred either as a result of the grain's self-ignition or due to arson." The conclusion, "In the grain store in question, either the rules for the storage of grain are not observed or it is not well guarded", is understood and not stated in explicit form.

2. *One of the premises is omitted from an inference*

It is possible to omit the first premise from an inference; it may be understood if it expresses a known proposition, theorem, law, etc.

In the conditional-categorical inference "The sum of the figures making up the said number is divisible by 3, and so the said number is divisible by 3", the first premise is omitted, which formulates the familiar law of mathematics: "If the sum of the figures in a given number is divisible by 3, then the entire number is divisible by 3."

The following disjunctive-categorical inference also omits the first premise: "Triangles may be acute, obtuse or right-angled", and the entire inference is formulated as follows: "The said triangle is neither acute nor obtuse. Consequently, the said triangle is right-angled".

In the example of a complex constructive dilemma given below: "If I go through the swamp, I may end up in the quagmire, and, if I go round it, I shall not manage to deliver the message on time. Consequently, I may end up in the quagmire or fail to deliver the message on time". The second premise is not formulated since it is understood: "I may go through the swamp or go round it".

It would be possible to give other examples of incomplete inference with the first or second premise omitted, but we will leave the reader to do this for himself.

Direct inferences we have examined, such as purely conditional, purely disjunctive, conditional-categorical, disjunctive-categorical and conditional-disjunctive (lemmatic), whether formulated in full or incomplete (i. e., in some of them either one of the premises or the conclusion is omitted) are widely used in the process of scientific and everyday thinking and in the teaching process. For this reason, a knowledge of the rules for constructing such inferences guards against logical errors in thought and helps to provide greater power of proof and conviction in the construction of arguments and approaches to the teaching of pupils and students.

Apart from the forms examined above, we also encounter the following direct inferences:

1. Simple contraposition.

The rule for a simple contraposition looks like this:

$$\frac{a \to b}{\bar{b} \to \bar{a}}.$$

This rule reads as follows: "If *b* follows from *a*, then the negation of *a* follows from the negation of *b*". Here *a* and *b* are variables denoting any proposition or propositional variable.

Example:

1. If the triangle in question is equilateral, then it is equiangular.

If the triangle in question is not equiangular, then it is not equilateral.

2. If this substance is phosphorus, then it does not combine directly with hydrogen.

If this substance combines directly with hydrogen, then it is not phosphorus.

Let us note that in the algebra of logic $\bar{\bar{a}} = a$

Formula $(a \to b) \equiv (\bar{b} \to \bar{a})$ is called the law of simple contraposition.

2. Complex contraposition.

$$\frac{(a \wedge b) \to c}{(a \wedge \bar{c}) \to \bar{b}}$$ is the rule of complex contraposition.

Example:

If I have the money and I am well, I shall go home for the holidays.

If I had the money and did not go home for the holidays, then I was consequently not well.

3. Rule of importation (conjunctive combination of conditions). This rule is also called the rule of combination of premises:

$$\frac{a \to (b \to c)}{(a \wedge b) \to c}.$$

This rule reads as follows: "If it follows from *a* that *c* follows from *b*, then *c* follows from *a* and *b*".

The Soviet educationalist V.A. Sukhomlinsky wrote: "If a teacher has become a pupil's friend, if this friendship is illuminated with a noble pursuit, with an inspired search for something sacred and sensible, then evil will never appear in the heart of the child." Basing ourselves on the rule for combining premises (rule of conjunctive combination of conditions), we may express Sukhomlinsky's proposition in a different way, but it will be equivalent to the original one. Conclusion: "If a teacher has become a pupil's friend and this friendship is illuminated with a noble pursuit, with an inspired search for something sacred and sensible, then evil will never appear in the heart of the child."

4. Rule of exportation (disjunction of conditions).

$$\frac{(a \wedge b) \rightarrow c}{a \rightarrow (b \rightarrow c)}.$$

This rule reads as follows: "If *c* follows (derives) from *a* and *b*, then it follows from *a* that *c* follows from *b*." This rule is the obverse of the previous one. For this reason, it may be illustrated by taking Sukhomlinsky's same proposition, but first reading the conclusion we obtained from which it is possible to arrive at Sukhomlinsky's original proposition.

§ 10. Indirect Inferences

These include: argumentation according to the rule of implication introduction; *reductio ad absurdum* and negative inference.

1. Argumentation according to the rule of implication introduction. The rule of inference is formulated as follows:

$$\frac{G, a \vdash b.}{G \vdash a \rightarrow b.}.$$

This rule reads as follows: "If *b* derives from a set of premises (G) and premise *a*, then it derives from the set of premises alone that *b* follows from *a*." This rule of inference is also called the *Theorem of Deduction*. Here, G may also be a null set of premises.

Let us give an example of human reasoning to illustrate this rule. Let G contain the following premises: (1) "I have bought a car"; (2) "I have obtained a driving licence"; (3) "I have some free time". Premise *a* denotes: "I have some money" and conclusion *b* denotes: "I shall take a holiday in the car with my family". What is written above the line will be read as follows: "If I have bought a car, obtained a driving licence, have some free time and some money, then the following conclusion may be inferred": "I shall take a holiday in the car with my family". What is written below the line may be read as follows: "I have bought a car, obtained a driving licence and have some free time". From this, the conclusion follows: "If I have some money, I shall take a holiday in the car with my family".

2. The *reductio ad absurdum* rule. This rule is also called the rule of introduction of negation.

$$\frac{G, a \vdash b;\ a \vdash \bar{b}}{G \vdash \bar{a}}.$$

The rule reads as follows: "If a contradiction, i.e., *b* and *not-b*, derives from premises G and premise *a*, then the negation of *a* derives from G alone". The method of reduction to the absurd is a widely employed mode of reasoning.

In classical two-valued logic, the method of *reductio ad absurdum* is expressed by the formula: $\bar{a} \underset{Df}{=} a \rightarrow F$, where F is a contradiction or a falsehood. This formula indicates that judgement *a* should be negated (considered false) if a contradiction derives from *a*.

The definition of negation by means of reduction to the absurd, to a contradiction, is widely employed in both classical and non-classical systems of logic, be they multi-valued, constructive or intuitionistic.

3. The rule of indirect (negative) inference. Negative proof is a method employed when there is no argument for direct proof. This method is often used to prove mathematical theorems.

The essence of negative inference will be illustrated in the part of the section on proof devoted to indirect proof.

We have thus examined the rules of direct and indirect inferences and seen that both types are used extensively in thought. We have shown how the various forms of direct and indirect proof are invested with concrete substance taken from the fields of educational science, mathematics, physics, ethics and other disciplines and areas of everyday thought as well as the practice of teaching.

§ 11. Inductive Inferences and Their Types

The logical nature of induction

Provided the appropriate rules are observed, deductive inferences allow true conclusions to be drawn from true premises. Inductive inferences may yield both true and probable (plausible) conclusions.

Induction is inference from a piece of knowledge with a lower degree of generality to a new piece of knowledge with a higher degree of generality (i. e., from the particular to the general).

In nature and society, the general does not exist independently, before and outside the particular, and the particular does not exist without the general; the general exists in the particular and through the particular, i. e., it manifests itself in actual objects. This is why the general and the essential, which repeats itself and represents a natural law of objects, is perceived through the study of the particular, and one way of perceiving the general is by means of induction, which is said to be complete, incomplete, and mathematical.

A *complete induction* is the name given to an inference in which a general conclusion about a class of objects is reached by studying all the objects in this class.

The conclusion may be made up of individual judgements, as can be seen from the example given below. The phenomenon referred to is what is called the "parade" of planets. Once every 179 years, all the planets line up at one side of the sun in a sector with an angle of about 95 degrees. The point of their closest approach was recorded on March 10, 1982 (although in total it is a long process taking several years).

In 1982, the Earth was located together with the other planets on one side of the Sun in a sector with an angle of about 95 degrees.
In 1982, Mars was located together with the other planets on one side of the Sun in a sector with an angle of about 95 degrees.

. . . .

. . . .

. . . .

In 1982, Mercury was located together with the other planets on one side of the Sun in a sector with an angle of about 95 degrees.
The Earth, Mars, Venus, Neptune, Pluto, Saturn, Uranus, Jupiter and Mercury are the planets of the solar system.

In 1982, all planets of the solar system were located on one side of the Sun in a sector with an angle of about 95 degrees.

A conclusion may be reached from complete induction not only using individual, but also universal judgements. Proof by instances is a case of complete induction.

Mathematics uses numerous cases of proof by instances. The following theorem is a case of proof from a selection of instances: "The volume of a rectangular parallelepiped is equal to the product of three of its measurements" ($v = a \cdot b \cdot c$). In proving this theorem, we may particularly examine the following three instances: (1) the measurements are expressed in whole numbers; (2) the measurements are expressed in fractions; (3) the measurements are expressed in irrational numbers.

Complete induction gives a necessary conclusion and is therefore employed in mathematical and other rigorous proofs. In order to employ complete induction, one has to fulfil the following provisos: (1) To exactly know the number of objects or phenomena to be studied. (2) To be certain that the property in question belongs to each of the objects in this class. (3) The number of objects in the class to be studied should be small.

A variety of complete induction is inference from individual parts to the whole. For example, the fulfilment of the Soviet Union's economic plan is determined by its fulfilment in each individual Union Republic.

Mathematical induction

One of the most important methods of proof in mathematics is based on the axiom (principle) of mathematical induction. Assume: (1) property A obtains when $n = 1$; (2) from the assumption that real number n possesses property A, it follows that number $n + 1$ also possesses property A. This leads to the conclusion that any natural number possesses property A.

Mathematical induction is used to derive a series of formulas of arithmetic and geometric progression, the formulas of Newton's binomial theorem, etc.

§ 12. Types of Incomplete Induction

Incomplete induction is used when we are unable to observe all instances of the phenomenon being studied, but draw a conclusion for them all. For example, we observe that, when heated, nitrogen, oxygen and hydrogen expand, we conclude therefrom that all gases expand on heating. One of the types of incomplete induction – scientific induction – is extremely important, since it allows the formulation of general principles and laws of science (for example, Ohm's law, the law of causality, social laws and others).

Incomplete induction comes in three different types, depending on their means of support.

Type I. Induction by ordinary recurrence (popular induction)

The recurrence of one and the same property in a series of homogeneous objects, and the absence of any examples to contradict this, serve as the basis for the conclusion that all objects of the said class possess this property. For example, it was popular induction that all swans were white, until black swans were encountered in Australia. This kind of induction provides a possible, but not necessary, conclusion. It is of a certain value in the initial stages of constructing a hypothesis, but later needs to be checked. A characteristic and extremely widespread error is "premature generalisation". For example, people who encounter errors in the evidence of witnesses on several occasions are inclined to say: "All witnesses err", or a pupil may be told, "You know nothing about this question", etc.

People have used popular induction to derive many useful omens: when the swallows fly low there will be rain; a red sunset is a sign of a windy day tomorrow, etc.

Type II. Induction by analysis and selection of facts

In popular induction, the objects to be observed are chosen at random, without any system. In induction by analysis and selection the coincidental nature of generalisation is ruled out, since the objects in question are selected and represent the most typical ones. They differ in terms of time, the means by which they are obtained and of their existence, as well as other factors. This procedure is employed, for example, to find out the average yield of a field, or the germinating capacity of seeds, or the quality of large batches of goods, or the composition of minerals which have been prospected. In studying the quality of a batch of tinned fish, sample tins, with different output dates and of various sorts, are taken from various refrigerators.

Even in ancient times, people saw from many years of observation that silver purifies drinking water. Silver salts were added to potions used to heal burns. People gradually reached the conclusion that silver possessed curative effect, a conclusion drawn on the basis of induction by selection. Subsequently, scientific investigations showed that silver activates oxygen, which then kills bacteria, and so the original conclusion turned out to be correct.

The concept of probability

A distinction is made between two types of the concept "probability", namely objective and subjective probability. An *objective probability* is a concept describing the quantitative possibility of some event occurring under definite conditions. This type of probability is used to describe objective properties and relations pertaining to mass phenomena of an accidental nature. Objective possibility is expressed using the mathematical theory of probability. For example, the probability of a coin falling "heads" up when tossed may be expressed as 1/2, and the probability of a certain number being obtained on the throw of a die is equal to 1/6. The concept of mathematical probability can only be fruitfully applied to mass phenomena, i. e., those which constantly recur. These include the birth of a baby with a definite sex, the appearance of a certain letter in a long text, rainfall, the occurrence of a faulty product in a series of mass-produced items, etc.

A *subjective probability* makes it possible to analyse the peculiarities of human subjective cognitive activity in conditions of indefiniteness. For example, a man may state, "It is extremely likely that in the next few years automatic manipulators (industrial robots) will become much more widespread in industry". Here, the likelihood being expressed is a measure of subjective confidence. The latter is determined, first, by the information someone has (or has not) at his disposal; second, by the individual's psychological peculiarities, which play an important role when he comes to assess the probability of some or other event occurring. We use various words to describe such phenomena in speech: "very probable", "hardly probable", "improbable", "unlikely", etc.

In order to raise the degree of probability of conclusions reached through induction by means of analysis and selection of facts, it is essential to observe the following provisos:

(1) the number of samples studied from the class in question should be sufficiently large. For example, to be considered representative, an opinion poll must be carried out among a certain percentage of people making up the group in question; in each concrete case, this percentage, this number of samples selected from the class of objects will be different;

(2) the samples must be selected according to a system and be as varied as possible;

(3) the property being studied, according to which the objects are classified, should be typical of all members of the class;

(4) the property being studied should be essential to the objects in the class.

Let us give examples from sociological surveys, including those carried out among youth.

The entire set of members of a social entity studied under the programme of sociological investigations within definite territorial and time limits, constitutes the general group. Total surveys are possible which represent cases of complete induction (for example, the nationwide Soviet censuses carried out by the Central Statistical Board in 1959 and 1979, or the study of all persons within the limits of a given region, city, institution, school, etc.). Here we are considering incomplete induction. An example of this is a concrete sociological investigation carried out by studying a certain part of the general group. The part of social objects from the general population which is observed is called the sample group. The model (i.e., sample group) is, of course, smaller in size than the modelled (general) group. In order to better study the whole, a more precise and correct choice must be made of the part to be studied, so that less mistakes are made in the conclusions drawn about the whole.

There are various types of sampling: random, quota, probable, etc. In this connection, the following demands must be taken into account: completeness, accuracy, adequacy, convenience of work, absence of duplication of units being observed. The basis may be formed by a list in alphabetical order of the staff of an institution, school, organisation, etc. For example, in a study of work satisfaction or the social activity of youth at a particular place of work, the basis for the sample will be formed by a list of the young people employed at this place of work.

The extent of sampling is the total number of units for observation as included in the sample group. Sampling should be sufficiently extensive; it depends on the level of homogeneity of the general group and the necessary degree of accuracy of the results. Sampling which is sufficient to study one property may be insufficient for another.

A mistake is often made in sampling by "choosing those like oneself". This error is often made by young interviewers from high schools who tend to interview those with whom they find it easier to communicate, which results in overstating the share of people with higher education or of young people.

Provided all conditions for the chosen type of sampling are observed, the degree of probability of conclusions induced by analysis and a selection of facts will be raised.

Type III. Scientific induction of causality

Scientific induction is inference in which cognition of the essential features or the essential relationship of a part of a class of objects is used to obtain a general conclusion about all objects in this class. Just like complete induction and mathematical induction, scientific induction yields a necessary conclusion. The necessity (and not probability) of conclusions derived by scientific induction, although it covers not all objects belonging to the class being studied, but only a part of them (and a small part at that), is ensured by account being taken of the most important of all essential links, namely the causal one. For example, scientific induction yields the conclusion: "All people need water to live." A man can live without food (starving completely) for 30 to 40 days, but he must have water every day: a human being cannot live without water, since the removal of water from the organism leads to disturbances of the internal metabolic processes, as a result of which the human being dies. Starvation, in contrast, when carried out under a doctor's control, is effective in the case of many illnesses (for example, chronic nephritis, high blood pressure, stenocardia, atherosclerosis, neurodermatitis, bronchial asthma, schizophrenia, obesity, and many others) if undertaken as a repeated course of treatment. This conclusion was obtained by scientific induction.

The reason for treating these diseases by prolonged starvation is the amazing self-regulation of the human organism during curative starvation, when it undergoes an overall biological readjustment. Habitual overeating, which imposes a huge and quite unnecessary strain on the stomach and the heart, is the main cause of many illnesses, fatigue, early senility and premature death.

Scientific induction was used to formulate all scientific laws (natural and social), including the physical laws of Archimedes, Kepler, Ohm and others. For example, Archimedes' law is a demonstration of the fact that every liquid exerts an upward pressure on a body immersed in it.

Scientific induction was used to obtain the laws of development of society, for example, the law that production relations correspond to the nature and level of development of the productive forces, or the fundamental sociological law defining the role of the mode of production in social development.

Scientific induction relies not so much on a large number of investigated facts as on the comprehensive way in which they are analysed and the establishment of causal dependence, the revelation of the essential features or the essential connections of objects and phenomena. For this reason, scientific induction yields a necessary conclusion.

The Soviet philosopher S. A. Lebedev has studied the development of the category "induction" from the science of antiquity to the present day. He singled out three stages in the development of the specific methodological problems: (1) induction and the problem of the origins of knowledge (science of antiquity); (2) induction and the problem of discovering and proving scientific laws and theories (modern age, 19th century); (3) induction and the problem of corroborating and adopting scientific hypotheses and theories (19th century).

An important direction in the development of the category "induction" was its bifurcation into method and inference (this was done by Aristotle, John Stuart Mill and Stanley Jevons). Induction as a method of scientific cognition is a complex substantive operation including observation, analysis, selection of material, experimentation and other means. By viewing induction as inference, it was possible to allocate to it a class of its own. Later, induction was divided into formal and material. Formal classical induction is any inference where premises are less general than the conclusion. The specific nature of the premises' content (everyday, concrete scientific, philosophical, etc.) is abstracted from. Material classical induction is inference from a less general piece of knowledge to one which is more general, but account is being taken of its content, a fact of substantial importance.

S. A. Lebedev writes that later material induction "was split into scientific and non-scientific. In contrast to non-scientific induction, scientific induction relies in its premises only on essential links and relations, as a result of which its conclusions are necessary, although in terms of logical form it may represent incomplete enumerative induction, that is a non-proven type of inference from the formal viewpoint".

The development of the category "induction" has seen not only quantitative changes (an increase in the number of types, forms and functions of induction), but also qualitative ones. Whilst in classical logic and methodology of science induction was understood as the movement of cognitive thought from a less general piece of knowledge to a more general one, in contemporary logic and methodology of science, the term "induction" is often used as a synonym for the concepts "non-demonstrative inference", "probabilistic argument", etc. Such are the systems of inductive logic devised by Rudolf Carnap, Ya. Hintikka and other logicians. They provide a logical explication of induction, not in its classical understanding, but in the sense of a non-demonstrative, probabilistic conclusion. This occurred because they identified one of the properties of classical inductive inferences, namely their non-demonstrative character, with the essence of inductive inferences. Lebedev rightly states that identification of the concepts "induction" and "inductive inference" with the concepts "probabilistic inference" and "non-demonstrative argument" is not an innocuous linguistic innovation, as it may seem at first sight. The range of problems pertaining to probabilistic inferences does not at all cover the major epistemological and methodological areas that were traditionally associated with the classical understanding of induction. Moreover, the essential difference between the classical and modern understanding of induction has not been duly reflected in correspondingly different terminology, which leads to obvious confusion in posing and solving a whole number of methodological problems, such as induction and the discovery of scientific laws, induction as a guide to life, the correlation between induction and deduction and other types of inferences and methods of scientific cognition, etc.

Since the classical understanding of induction differs from the contemporary one and both are equally legitimate, it is desirable to avoid confusion by using different terms, i. e., by calling induction in the classical sense "induction_1" and in the modern sense "induction_2".

This introduction of two different terms is doubtless to be welcomed, since previously there were violations of the law of identity, so that the term "induction" was used in two different senses.

We feel that the same applies to the term "deduction". Here, it is also necessary to introduce two terms: "deduction_1" (classical deduction, i. e., inference from a more general to a less general notion) and "deduction_2" (the contemporary understanding of deduction as a necessary inference or demonstrative inference, i. e., one which, provided certain introduced rules are observed, yields a conclusion which necessarily follows from the premises). Here, too, substitution occurred, i. e., the law of identity was violated, since the term "deduction" was used in different senses. This leads to confusion and a lack of mutual understanding when one person uses the term "deduction" in the classical sense and another in its contemporary interpretation, as is accepted practice in symbolic logic.

§ 13. Inductive Methods of Establishing Causal Connections

The concept of cause and effect

A *cause* is a phenomenon or a totality of phenomena which give rise to another phenomenon (effect).

A causal link is universal, since all phenomena, even accidental ones, have their cause. Accidental phenomena are subject to the laws of probability, i. e., the laws of statistics.

A causal link is a necessary one, since an action (effect) will necessarily follow in the presence of the cause. For example, good training and musical abilities are the cause of someone becoming a good musician. But causes must not be confused with conditions. A child may be provided with all the necessary conditions, with an instrument and sheet music, a teacher and books on the subject, etc., but if he does not have the talent he will not make a good musician. Conditions facilitate or, on the contrary, hamper the action of the cause, but conditions and cause are not identical.

Methods of establishing causal connections

A causal link between phenomena may be defined by a number of methods which were described and classified by Francis Bacon and developed by John Stuart Mill.

Similarity method. Let us suppose we have to discover the cause of some phenomenon *a*. Basing ourselves on the definition of cause as a phenomenon or a totality of phenomena preceding some other phenomenon and giving rise to it, in this case phenomenon *a*, we shall analyse the phenomena which preceded *a*. In the first instance of the appearance of phenomenon *a*, it was preceded by circumstances *ABC*, in the second instance *ADE* and, in the third instance, *AKM*. What could be the cause of *a*? Since in all three instances, the common circumstance was *A*, and all other circumstances were different, we may conclude that *A* is probably the cause or part of the cause of phenomenon *a*.

Instances of appearance of event *a*	Preceding circumstances	Phenomenon observed
1	*ABC*	*a*
2	*ADE*	*a*
3	*AKM*	*a*

A is probably the cause of *a*.

An example of the single similarity method is the clarification of the cause of three people becoming ill with encephalitis. In the first instance of a person becoming ill with encephalitis, the onset of the disease was preceded by the following events: *A* – bite by the

Ixodes tick; B – beginning of the summer period; C – stay in the woods in the Urals. In the second instance, the illness was preceded by the following events: A – bite by the Ixodes tick; D – spring period, E – stay in a wooded area of Eastern Siberia. In the third instance, the illness was preceded by the following circumstances: A – bite by the Ixodes tick; K – end of the summer period; M – stay in a birchwood forest in the Altai region. The common feature in all three cases of people becoming ill with encephalitis was a bite by the Ixodes tick, which was the possible reason for their illness.

If the observed cases of any phenomenon have just one common circumstance, then it is obviously the cause of the said phenomenon.

The method is associated with observation.

The *distinction method* is used when we examine two instances and note that in the first case *a* occurs and in the second case *a* does not occur. A study of the preceding circumstances shows that in both instances they coincide except for one circumstance which was present in the first instance and absent in the second.

Instance	Preceding circumstances	Phenomenon observed
1	*ABCD*	*a*
2	*BCD*	–

A is probably the cause of *a*.

The distinction method is more associated with experimentation than with observation, since we have voluntarily to single out one or another circumstance. For example, to find out at an airport whether any passengers are carrying any large metal objects, they are made to pass through a device fitted with an electromagnet and an electric bell connected to it. When a passenger passed through the device, the bell rang. He was asked to take all metal objects out of his pockets. When he removed his key ring and coins and passed a second time, the bell did not ring. Consequently, the cause of the bell ringing was the presence of the said metal objects. All other preceding circumstances were the same.

If instances in which a phenomenon occurs or does not occur only differ in one preceding circumstance, and all other circumstances are identical, this one circumstance is the cause of the phenomenon in question.

Here is another example. If someone eats strawberries and has an allergic reaction afterwards, whilst all other food products were the same, and if on the days after he had not eaten strawberries and had no allergic reaction, the doctor will conclude that the strawberries he ate caused the allergy in the said patient.

Accompanying changes method. If a change in preceding circumstance A causes a change in phenomenon a we are studying, while all other preceding circumstances (for example B, C, D and E) remain unchanged, then A is the cause of a.

For example, if we double the velocity of motion, then the distance covered in the same time will also double. Consequently, an increase in velocity is the cause of an increase in the distance travelled over the same time interval.

$S = v \cdot t$ is the formula of uniform motion and establishes that, given a change in v or t (velocity of motion and time of motion), there will be a directly proportional change in the distance (S).

Friction is the cause of bodies becoming hot; an increase in the length of a metal rod indicates that it has become hot. These and other examples illustrate the use of the accompanying changes method. However, we cannot separate friction from heating, since we could not apply the distinction method to establish the cause of the body becoming hot.

If a change in one circumstance always causes a change in another, then the first circumstance is the cause of the second.

Remainder method. Let us assume that phenomenon K being studied is split into several homogeneous parts: a, b, c, d. It has been established that it was preceded by circumstances A, B and C. It is also known that A is the cause of a, B is the cause of b and C is the cause of c. Circumstance D which is the cause of the remaining unexplained phenomenon d must be similar to A, B and C.

An example which illustrates this method is the discovery of the planet Neptune. Observing the extent to which the planet Uranus deviated from its calculated orbit, scientists came to the conclusion that the deviations to extents *a, b* and *c*, caused by the presence of planets *A, B* and *C*, were not sufficient to account for the full deviation from the calculated orbit. There remained value *d*. On the basis of this, it was concluded that there must be an unknown planet *D* causing this deviation. Urbain Le Verrier calculated the location of this unknown planet and, having built a telescope, Johan Gottfried Galle found it in the skies in 1846. That is how Neptune was discovered.

If it is known that the cause of a phenomenon being studied is not constituted by any of the circumstances necessary for it except one, then this one circumstance is probably the cause of the said phenomenon.

The examined methods of establishing causal links are more often than not employed not in isolation, but in combination, so that they complement each other.

§ 14. Deduction and Induction in the Teaching Process

Just like in any other thought process (be it scientific or everyday thought), deduction and induction are interlinked in the teaching process.

In induction, we proceed from premises expressing a piece of knowledge of a lesser degree of generality to a new piece of knowledge with a greater degree of generality, from individual concrete phenomena to generalisation. In deduction, the course of reasoning is the opposite, i.e., we proceed from generalisations and inferences to concrete facts or judgements with a lesser degree of generality.

In the teaching process, the inductive and deductive methods are used integrally. The inductive method is used when studying new material which the pupils find difficult and when, as a result of a discourse, they are able themselves to draw a definite conclusion, formulate a rule, theorem or some law. The inductive method greatly promotes the pupils' activity, but demands a creative and flexible approach on the part of the teacher. The process of leading the pupils to their own conclusion is more time-consuming.

The deductive method is employed when the teacher himself formulates a general judgement expressing some rule, law, theorem, etc., and then uses it, providing illustrations by means of particular examples, instances, facts, events, etc. The conjunction of induction and deduction in the teaching process makes it possible to explain the relevant material in two ways: the *inductive-deductive way* of exposition, beginning with induction and then proceeding to deduction (possibly with a considerably greater emphasis on induction), and the *deductive-inductive way*, when the teacher himself provides the pupils with a new piece of knowledge as a ready-made, formulated rule or proposition and then goes on to comment.

The famous Russian educationist K. D. Ushinsky set great store by induction in the study of grammar. Using specially selected examples, he developed in children an ability to perceive the rules of language and make their own generalisations, a factor of enormous importance in developing thought on the part of younger schoolchildren. Ushinsky attached no less importance to deduction and accorded a major role in the study of language to exercises whereby pupils themselves searched for examples to illustrate laws which had just been formulated. These approaches are used not only in lessons in the native language, but also in mathematics, history, physics and other lessons. In line with the existing methodology, school teachers are recommended to use these methods in a more concrete way when working on individual themes from the curriculum.

There are many devotees of both the inductive and deductive methods in mathematics. Some mathematicians believe that preference should be given to the inductive method in the initial stages of study, and that preparations should be made for the gradual introduction of the deductive method, since the inductive method of exposition, involving the generalisation of concepts, facilitates more active assimilation of the material in question. In recent years a tendency is observed to oust the inductive method for the benefit of the deductive one, and the expedience of this often seems doubtful.

However, both inductive and deductive methods demand that considerable time be given to concrete illustrations, the analysis of examples and concrete situations when new concepts or new general theories are introduced. It rests on the teacher to find an optimal choice of methods enabling the organisation of pupils' independent cognitive activity.

Various types of induction are used in mathematics, namely complete, incomplete and mathematical. The use of mathematical induction may be shown by the following example. The object is to define the sum n of the initial odd numbers:

$$1 + 3 + 5 + 7 + \ldots + (2n - 1).$$

If we denote this sum as $S(n)$ and let $n = 1, 2, 3, 4, 5$; we will obtain:

$$\begin{aligned} S(1) &= 1, \\ S(2) &= 1 + 3 = 4, \\ S(3) &= 1 + 3 + 5 = 9, \\ S(4) &= 1 + 3 + 5 + 7 = 16, \\ S(5) &= 1 + 3 + 5 + 7 + 9 = 25. \end{aligned}$$

We notice an interesting rule: when $n = 1, 2, 3, 4, 5$, the sum n of consecutive odd numbers is equal to n^2. But we cannot draw a conclusion by analogy and say that this applies for any n, since the conclusion may be false. Let us use the method of mathematical induction, i.e., let us suppose that for some number n our formula is correct and try to prove that it is then true for the following number $n + 1$. So we suppose that $S(n) = 1 + 3 + 5 + \ldots + (2n - 1) = n^2$. Let us calculate $S(n + 1) = 1 + 3 + 5 + \ldots + (2n - 1) + (2n + 1)$. But, according to our assumption, the sum n of the initial addends is equal to n^2, so that:

$$S(n + 1) = n^2 + (2n + 1) = (n + 1)^2.$$

Thus, assuming that $S(n) = n^2$, we have proven that $S(n + 1) = (n + 1)^2$. And we checked above that this formula is correct for $n = 1, 2, 3, 4, 5$, which means that it will be correct for $n = 6$ and $n = 7$, etc. The formula is thus considered proven for any number of addends.

The same method is used to prove that the sum n of the initial real numbers, denoted as $S_1(n)$, is equal to $\frac{n(n+1)}{2}$, i.e., $S_1(n) = 1 + 2 + 3 + 4 + 5 + \ldots + n = \frac{n(n+1)}{2}$.[1]

In mathematical thought, we have not only logical reasoning, but also mathematical intuition, fantasy and a feeling of harmony, which make it possible to envision the course by which a task can be resolved or a theorem proved. However, in mathematics it is for cold reason to study, prove or disprove intuitive considerations and verisimilar arguments. The truth of a judgement is not proven by a number of examples, and not by a series of experiments, which has no power of proof in mathematics, but in a purely logical way, in conformity with the laws of formal logic. In the course of studying mathematics, it is supposed that knowledge, the mathematical apparatus, intuition, a feeling for harmony, fantasy, an ability to think, logic and experiments are being employed not consecutively and by stages, but all these factors interact to form an integral process. This interaction leads to the formation of mathematical culture among those studying in higher and secondary schools. Thus, the unity of deduction and induction in study and scientific creativity uniquely and vividly manifests itself in mathematics, a discipline which differs from the natural and social sciences both in the methods of proof it employs and the methods by which knowledge is passed on to students.

We described above types and gave examples of incomplete inferences (categorical syllogism, conditional, disjunctive inferences, etc.).

In the process of studying mathematics, students gain the ability to curtail the process of mathematical reasoning in solving problems of a type that is familiar to them. If they have repeatedly to solve problems of the

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[1] Readers interested in the use of induction in mathematics are recommended to read George Polya's book *Mathematics and Plausible Reasoning*, Vol. I. Induction and Analogy in Mathematics, Princeton, 1954.

same type, students omit certain stages of the thought process in their reasoning and cease to be conscious of them, but they can return to reasoning in all its stages if and when necessary. Methodologists in the field of mathematics have established on the basis of algebraic and arithmetical material that "alongside full-scale inferences, the mental activity of pupils in the solution of problems also deals with curtailed inferences when the pupil is unaware of the rule of the general proposition which he actually follows ... does not construct the entire chain of propositions and inferences which make up the complete, developed system of solution".[1] The shortening of the reasoning process in mathematics occurs as a result of habitual operations, with gifted learners quickly resorting to such techniques, medium learners adopting them more slowly, and slow learners failing to adopt such techniques despite repeated exercise. The Soviet psychologist V. A. Krutetsky puts forward the hypothesis that probably never and nowhere does man think in full-scale structures. However, gifted learners do think in curtailed structures and incomplete inferences not only in solving familiar problems, but also when faced with new ones; at the request of the experimenter, these learners would restore the curtailed structure to a complete one (as they see it). Curtailed thought structures facilitate a more rapid processing of information and speedier solution of problems and simplify the execution of complex operations.

In studying the components of the structure of mathematical abilities among school-children, Krutetsky analysed the statements made on this score by a number of Soviet mathematicians and mathematics teachers. Some 38 per cent of the polled noted that gifted learners used curtailed thought process. The following statements were recorded: "The process of reasoning is curtailed in gifted learners and never developed to the complete logical structure. This is extremely economical, and that is what makes it valuable"; "I have often noticed how a gifted student thinks – for the teacher and the class it is a chain process consecutive in all its links,

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[1] V. A. Krutetsky, *The Psychology of Mathematical Abilities*, Moscow, 1968, p. 291 (in Russian).

but for himself the process is fragmentary, rapid, curtailed, like a shorthand of thought".[1]

Listing the intellectual qualities of these students, almost all mathematics teachers and mathematicians polled (98 per cent) noted their *ability to generalise*. A gifted pupil quickly generalises not only the mathematical material, but also the method of reasoning, of proof. Some of the poll participants spoke of an ability and even singular "passion" for generalisation, an ability to "see what is general to different phenomena", "an ability to arrive at the general from the particular".

When called upon to use deduction and induction in non-mathematical subjects, the vast majority of senior students are able to select material pertaining to the theme of an essay, to generalise it, develop the fundamental idea behind the essay, determine the scope of the theme, and draw independent conclusions. But in some students an ability to apply a deductive line of reasoning is insufficiently developed. Having given a correct definition, they do not always cope with an analysis of a concrete instance from the viewpoint of this definition, so that there are no conclusions on the theme of the essay and there is a gap between their factual and theoretical knowledge.

The positive aspects and shortcomings in students' knowledge illustrated above indicate that it is important to achieve a judicious combination between induction and deduction in the teaching process. However, no universal recipes exist dosing this or that method in teaching. In this connection, it is interesting to note the viewpoint taken by the Soviet mathematician L. D. Kudryavtsev on the methodological principles of mathematics teaching: "Unfortunately there are no precise recipes on how to teach different divisions of mathematics. The methodology of mathematics teaching is not a science but an *art*. Admittedly, this does not mean that there is no point in teaching the methodology of mathematics instruction. Every art can and should be taught: visual artists, musicians, performers and writers all need to study."

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[1] *Ibid.*, p. 207.

One can use typical errors made in the teaching process to once again draw the conclusion that the various teaching methods should be used creatively and that a stereotyped approach is inadmissible.

§ 15. Inference by Analogy and Its Types. Use of Analogies in Teaching

The term "analogy" is used to denote the similarity of two objects (or two groups of objects) in terms of some qualities or relations. Inference by analogy is one of the oldest types of inference and has been characteristic of human thought from the very earliest stages of development.

Analogy is inference attributing a certain feature (i. e., property or relation) to an object on the basis of the similarity of its essential features with those of another object. This type of inference has a quality attributed to an object or relations transferred to it.

Depending on the nature of the information transferred from one object to the other (from the model to the prototype), the analogy may be one of two types: an analogy of properties or of relations.

In an *analogy of properties*, we consider two individual objects (or two sets of homogeneous objects, two classes), and the properties of these objects are the features which are transferred.

The pattern of an analogy of properties is the following in traditional logic:

Object A possesses properties a, b, c, d, e and f.
Object B possesses properties a, b, c and d.

Object B probably possesses properties e and f.

An example of an analogy of properties may be the emergence of the symptoms of some illness in two different people (two individual objects) or two different groups of people (for example, adults and children). Proceeding from the similarity of the features of the illness (symptoms), the doctor makes his diagnosis by analogy.

An analogy of the properties of two objects sometimes yields not only a possible, but even a necessary conclusion. For example, it was observed that the geological structure of the plateau in the south of Africa has a great deal in common with the East Siberian platform. A bluish mineral was found in the diamond veins of Southern Africa. Quite coincidentally the same bluish mineral was also found at the mouth of one of the rivers in Yakutia. The conclusion was drawn by analogy that there were probably diamond deposits in Yakutia as well. This conclusion was corroborated. Nowadays diamonds are mined on an industrial scale in Yakutia.

In an *analogy of relations*, the information transferred from the model to the prototype describes the relations between the two objects. Let us take relation (aRb) and relation (mR_1n). The similarities (analogies) are R and R_1, but a is not analogous to m, and b is not analogous to n. An example of this is the model of the planetary structure of the atom proposed by Rutherford on the basis of an analogy between the sun and planets, on the one hand, and the nucleus of an atom and its electrons, which are kept in their orbits by the attractive force of the nucleus, on the other. Here, R is the interaction of forces of opposite moment – forces of attraction and repulsion – between the planets and the sun, and R_1 is the interaction of forces of opposite moment – forces of attraction and repulsion between the nucleus of the atom and the electrons, but the planets are not analogous to electrons, and the sun is not analogous to the nucleus of an atom.

Apart from dividing them into two types – analogy of properties and analogy of relations – inferences may be divided into three types by the nature of the piece of knowledge obtained (the level of necessity of the conclusion): (1) *rigorous* analogy (giving a necessary conclusion); (2) *non-rigorous* analogy (giving a possible conclusion); (3) *false* analogy (giving a false conclusion).

Rigorous analogy. A characteristic feature distinguishing a rigorous analogy from a non-rigorous or false analogy is the presence of a necessary link between the properties of similarity and the property being transferred. A rigorous analogy looks like this:

Object A possesses properties a, b, c, d and e.
Object B possesses properties a, b, c and d.
From the totality of properties a, b, c and d, there inevitably follows e.

Object B inevitably possesses property e.

If e necessarily follows from the totality of features $M = \{a, b, c, d\}$, this dependence is written as follows as a formula of the algebra of logic:

$$(a \wedge b \wedge c \wedge d) \rightarrow e.$$

This formula is a law of logic, since by definition the logical consequent e cannot be false [i.e., property e is absent], when the premises are true [i.e., there is the totality (conjunction) of properties of similarity a, b, c and d]. Like the structure of the *modus ponens* rule of conditional-categorical inference, the structure of a rigorous analogy thus gives a necessary, and not a possible, conclusion. This difference between them lies in the fact that in *modus ponens* there is always one antecedent and one consequent, whereas in a rigorous analogy there is a single totality of antecedents (similar properties) taken as a single set (not empty and not individual). If the set were empty, i.e., there were no similar properties, an analogy would be impossible, and if the set were individual, it would be *modus ponens*, which is expressed by the formula $((a \rightarrow b) \wedge a) \rightarrow b$.

Rigorous analogies are used in scientific research and when providing mathematical proof. For example, the formulation of the like features of a triangle is based on rigorous analogy. "If the three angles of one triangle are equal to the three angles of another, then these triangles are similar" (similarity is a type of analogy).

The modelling method is based on the properties of inference by rigorous analogy. Scientific analogies enable use to be made of the revolutionary experience of other countries and parties. In this connection, apart from considering the formal logical principles for drawing analogies, it is also necessary to take into account the methodological demand for the concreteness of truth and to examine a phenomenon in its concrete historical context.

If they are able to govern a thermonuclear reaction and thus create thermonuclear energy, people will obtain a practically unlimited source of fuel. High-temperature plasma in natural form exists in the atmosphere of the stars. Academician Yevgeni Velikhov writes that controllable thermonuclear fusion calls for a solar substance which does not normally occur under terrestrial conditions, namely hydrogen plasma at a temperature of about 100 million degrees (C). On the sun, it is held by the gravitational field, whilst on earth it may be kept under control by a magnetic field. However, a magnetic field is not at all like a gravitational one in terms of its impact on plasma particles.

Soviet scientists found a solution to this problem. The theory of the equilibrium and stability of plasma which they developed is already applied in the design of thermonuclear devices, including those of the T-15 type. A rigorous analogy between the thermonuclear processes taking place on the sun and those on earth makes it possible to substantiate ways of seeking solutions to controllable thermonuclear fusion.

Rigorous analogy provides a necessary conclusion, i.e., a truth denoted by 1 in classical logic, many-valued logics and the theory of probabilities.

Non-rigorous analogy

In contrast to a rigorous analogy, a non-rigorous analogy does not provide a necessary, but just a possible, conclusion. If we denote a false judgement by 0 and a true judgement by 1, then the degree of probability of conclusions reached by non-rigorous analogy lies somewhere between 1 and 0, i.e., $1 > P > 0$, where P denotes the probability of a conclusion reached by non-rigorous analogy.

The following represent examples of non-rigorous analogy: the testing of a model of a ship in a tank and the conclusion that the actual ship will exhibit the same characteristics; the testing of the strength of a bridge on the basis of a model and the construction of the actual bridge. Provided that all the rules for constructing and testing a model are strictly observed, this means of inference may approach a rigorous analogy and provide a necessary conclusion, although the conclusions reached in this way are more often than not possible. The difference in scale between the model and the prototype (the actual construction) is sometimes not only quantitative but also qualitative. It is not always possible to consider the differences between the laboratory conditions under which a model is tested and natural conditions, which leads to mistakes.

The following conditions should be observed in order to raise the degree of probability of a non-rigorous analogy:

(1) the number of common features should be as large as possible; (2) the similar features should be the essential ones. An analogy based on the similarity of non-essential features is typical of non-scientific thought and thought in children. For example, children may eat poisonous berries on account of their external resemblance to edible ones; (3) the common features should be as heterogeneous as possible; (4) account should be taken of the number and significance of the points of difference. If the objects differ in their essential features, then the conclusion by analogy may be false; (5) the feature being transferred should be of the same type as the similar features.

False analogy

If the above rules are violated, an analogy may yield a false conclusion, i. e., become a false one. The probability of a conclusion reached by a false analogy is equal to 0 ($P = 0$). False analogies are sometimes made deliberately in order to confuse an opponent, in which case they are an example of a sophistic approach. But in other cases, they are made accidentally, due to ignorance of the rules of constructing analogies or the lack of actual knowledge about objects A and B and their properties, which form the basis for the analogy.

This mistake was made in the 19th century by Ludwig Büchner, Karl Vogt and Jakob Moleschott, proponents of vulgar materialism who, drawing an analogy between the liver and the brain, declared that the brain secretes thought in the same way as the liver secretes bile.

Let us generalise what we have said about rigorous, non-rigorous and false analogies. If $P = 1$, i. e., if we obtain a necessary conclusion, the analogy will be a rigorous one. If $1 > P > 0$, i. e., the conclusion is possible, then we have an example of non-rigorous analogy. And when $P = 0$, i. e., when we have a false conclusion, the analogy will also be false.

We have thus examined three kinds of analogy in terms of the conclusion they yield, i.e., according to its degree of probability: we may obtain a true conclusion, a conclusion with a certain degree of probability or a false conclusion. Probabilistic conclusions are the more valuable the closer their probability is to 1 (true).

We suggest that this division into three types depending on the certainty of the conclusion may be followed up in inductive inferences and, later, in the section on hypotheses.

Among *inductive inferences*, we distinguish between: (1) *rigorous induction*, which gives a necessary conclusion and whose types are mathematical induction, complete induction and scientific induction; (2) *non-rigorous induction*, which gives a possible conclusion. It includes popular induction, induction by analysis and choice; (3) *false induction*, which gives a false conclusion that may lead to a logical error – "premature generalisation".

In the section on hypotheses, we shall see that there exist hypotheses which form a scientific theory and whose probability is equal to 1; the degree of probability of other hypotheses is $1 > P > 0$; the probability of false hypotheses, which are false judgements or false theoretical constructs, is equal to 0 (i.e., $P = 0$).

The situation is somewhat different in the case of many-valued logics, the simplest of which is three-valued logic. In some of them, truth is denoted by 1, falsehood by 0 and probability by 1/2 (the logics of Lukasiewicz, Heyting, Reichenbach and others are three-valued). These logics may be used to study the probability of conclusions by analogy, induction and also probability in hypotheses. Analogy, induction and hypothesis are in turn each divided into three types according to whether they yield a true, possible or false conclusion. Probability and indefiniteness are not one and the same thing. If indefiniteness has a fixed value, say 1/2, then the value of probability fluctuates between 1 and 0 ($1 > P > 0$), not including these actual values.

These three instances may be written as follows: $1 \geqslant P \geqslant 0$ (first instance, $P = 1$; second instance $1 > P > 0$; third instance $P = 0$).

The second instance ($1 > P > 0$) may be expressed in the infinite-valued logic G_{χ_0} (see Chapter VIII) devised by the author, not including its extreme values: 1 – truth and 0 – falsehood. The infinite-valued logic G_{χ_0} has the truth values $1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{7}{8}, \frac{1}{16}, \frac{15}{16}$ ..., 0 and includes all three instances embodied in the formula $1 \geqslant P \geqslant 0$. The process of cognition proceeds from ignorance to knowledge, incomplete, imprecise knowledge to more complete and precise knowledge, from relative to absolute truth, from a phenomenon to an essence of the first order, then to an essence of the second order, etc. It is an endless process of cognition and finds its reflection in the infinite-valued logic G_{χ_0}.

The use of analogies in teaching

Extremely broad use is made of analogies. We shall just give a few examples of analogies as employed in the teaching of physics, astronomy, biology and mathematics.

Many analogies are encountered in science, some of which give true conclusions and others false ones. The latter type are constructed in violation of the rules of logic. An example of a false analogy is that drawn by Herbert Spencer, who distinguished between various administrative organs in a class society and considered their functions analogous to those of the organs of a living body.

False analogies are frequently encountered in everyday thought. For example, some people believe that a broken mirror is an ill omen, or if you dig a knife in a stuffed animal before the hunt, the hunt will be successful.

Physics lessons generally make use of rigorous analogies which give true conclusions. It is well known that the unity of nature manifests itself in a "striking similarity" of differential equations referring to various areas of phenomena. Such analogical phenomena are extremely frequent in physics. As an example, we may give the corpuscular and wave-like properties of light and analogous properties of electrons. Another example is Coulomb's law, which defines the force of electrostatic interaction between two point charges q_1 and q_2 that are immobile vis-à-vis one another. The distance r between them is expressed by the formula

$$F = k \cdot \frac{q_1 \cdot q_2}{r^2}.$$

This law is formulated as follows: "The force of attraction or repulsion between two immobile point electric charges is directly proportional to the product of the charges and inversely proportional to the square of the distance between them". The coefficient k depends on the choice of units and the properties of the medium in which this interaction of charges takes place.

A similar formula is used to express Newton's universal law of gravity:

$$F = \gamma \cdot \frac{m_1 \cdot m_2}{r^2}.$$

Between two bodies with masses m_1 and m_2 that are located at a distance r from each other, there act equal forces of mutual attraction whose strength is directly proportional to the product of the masses and inversely proportional to the square of the distance; γ is the coefficient of proportionality, or the gravitational constant.

Here we have a rigorous analogy where the features transferred are *not properties but relations* between different objects (electric charges and masses of substance), expressed in formulas of analogous structure.

The teacher of physics also demonstrates in class the application of non-rigorous analogies, which are numerous in physics. Old ideas are frequently resorted to in the creation of new technologies. For example, a return is currently apparent to the use of sailing ships and airships. However, old ideas are being used at a new level and the only possible analogy is therefore a very distant one. In the past 20 years, communications satellites have increasingly replaced underwater cables, but it is planned to lay a new underwater cable for communications between Europe and America, this time based on fibre optics. In the latest watches, there again appeared a face with hands (and not with digital indication). However, these are not the hands to which we became accustomed, but an electronic analogy, a changing electronic image of hands.

The use of analogies in the teaching process is also characteristic of astronomy lessons. This is how George Polya describes Galileo's invention: "With his newly invented telescope, he discovered the satellites of Jupiter. He noticed that these satellites circling the planet Jupiter are analogous to the moon circling the earth and also analogous to the planets circling the sun. He also discovered the phases of the planet Venus and noticed their similarity with the phases of the moon."[1]

There are certain peculiarities to the use of analogies in mathematics. Only a rigorous analogy can be used as a means of proof. The functions of a non-rigorous analogy are much more varied. It is used especially often to solve problems of one type. Arithmetical, algebraic and geometrical problems are divided into types, species and subspecies and may be solved either by a certain algorithm or by analogy with other already solved problems (examples are the operations carried out with basins, motion, the construction of equations, geometrical problems with or without the use of trigonometry, etc.).

In mathematics an analogy is used when, trying to solve a problem we have been given, we seek another, simpler one. For example, in order to solve a problem in stereometry we find a similar problem in plane geometry: to establish the diagonal of a regular parallelepiped, we refer to a problem on the diagonal of a rectangle.

There may be a single-valued correspondence (called an isomorphism) between the elements of two systems S and S'. If we take S to be the sides of a rectangle and S' the faces of a regular parallelepiped, then the isomorphism (analogy) between them will be found in the fact that the relations between the sides of a rectangle are similar to those between the faces of a regular parallelepiped: each side of a rectangle is parallel and equal to one of the other sides and perpendicular to the others, and each face of a parallelepiped is parallel and equal to one of its other faces and perpendicular to the others.

We can construct analogies between other figures which are both plane and three-dimensional: between a triangle and a pyramid, a parallelogram and a prism. In mathematics teaching, analogies are also used in the following function. A theorem is provided: "The four diagonals of a parallelepiped have a common point which is the midpoint of each". The question is then

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[1] George Polya, *Mathematics and Plausible Reasoning*, Vol. I, Induction and Analogy in Mathematics, p. 26.

posed: "Is there a simpler analogous theorem?" The same question is also posed about the following theorems: "The sum of any two face angles of a trihedral angle is greater than the third face angle" and "If two straight lines in space are cut by three parallel planes, the corresponding segments are proportional."[1] Pupils are given tasks to select not only analogous theorems, but also analogous concepts: "Consider a tetrahedron as the solid that is analogous to a triangle. List the concepts of solid geometry that are analogous to the following concepts of plane geometry: parallelogram, rectangle, square, bisector of an angle."[2]

In geometry there is an analogy between a circle and a sphere. There exist two analogous theorems: "Of all plane figures of equal area, the circle has the minimum perimeter" and "Of all solids of equal volume, the sphere has the minimum surface". George Polya continues: "nature itself seems to be prejudiced in favor of the sphere. Raindrops, soap bubbles, the sun, the moon, our globe, the planets are spherical, or nearly spherical."[3]

Polya also draws an amusing example from the field of biology: when a cat is preparing to sleep on a cold night, it draws in its paws, curls up and in this way makes its body as close to a sphere as possible, evidently to retain warmth and cut losses through the surface of its body to a minimum. Polya continues: "The cat, who has no intention of decreasing his volume, tries to decrease his surface. He solves the problem of a body with given volume and minimum surface in making humself as spherical as possible."[4]

This analogy may be used in both mathematics and biology lessons. Analogies of relations may also be formulated, which are the basis for conclusions in bionics. Bionics is the science which studies objects and processes in living nature with a view to using the knowledge thus obtained in state-of-the-art technology. Let us give three examples. A bat in flight emits ultra-high frequency waves and then picks up their

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[1] George Polya, *Op. cit.*, p. 25.
[2] *Ibid.*
[3] *Ibid.*, pp. 169-170.
[4] *Ibid.*, p. 170.

reflection from objects so as to faultlessly find its way in the dark. It goes round the objects so as to avoid colliding with them and finds those it requires, such as insects or a place to roost, etc. Man used this principle to create radar, which is able to observe and determine the position of objects in any weather conditions. Vehicles have been developed to move through the snow, with their principle of motion borrowed from penguins. Using an analogy to the way in which jellyfish pick up infrasound with a frequency of 8 to 13 cycles per second (which enables jellyfish to sense the approach of a storm by the latter's infrasounds), scientists developed a piece of electronic equipment which can forecast a storm 15 hours in advance. The examples given in this section, and many others besides, will be useful in revealing to students the heuristic functions of inferences by analogy.

Exercises on inferences

I. You are given three premises: (a) if a whole number ends in 0 or 5, then it is divisible by 5; (b) the said number is divisible by 5; (c) the said number does not end in 0. Does it logically follow from these premises that the said number ends in 5?

II. Construct direct inferences – reduction, conversion and predicate opposition – from the following judgements: (a) "All extended sentences have secondary elements"; (b) "Some subjects are expressed by pronouns in the nominative case"; (c) "No student in our group is a chess player"; (d) "Some pilots are Soviet cosmonauts".

III. Prove in three ways: according to the special rules of figures, by modi and by the rules of categorical syllogism – whether the categorical syllogisms given below are valid and whether the conclusion is a true judgement.

1. All wolves are predators.
This animal is a predator.
———————————
This animal is a wolf.

2. All cinema halls need ventilation.
This room is not a cinema hall.
———————————
This room does not need ventilation.

3. All acts of theft are punished by law.

Driving away a car is an act of theft.

Driving away a car is punished by law.

4. All metals are solids.
Mercury is a metal.

Mercury is a solid.

IV. Form the full categorical syllogism from the following enthymemes.

1. All animals of the order *Proboscidea* are mammals since all proboscideans suckle their young.

2. Officials are obliged within prescribed periods to examine citizens' suggestions and petitions, to reply to them and take the necessary measures, and V. S. Petrov is an official.

3. All coniferous trees require moisture, and so a fir tree also requires moisture.

V. Define the type of inference, construct the pattern, write down the formula and prove that it is identically true.

1. A noun is an independent part of speech.
A proper name is a noun.
The name of a town is a proper name.
The name of a capital is the name of a town.
"Rome" is the name of a capital.

"Rome" is an independent part of speech.

2. Hydrocarbons are organic compounds.
Methane is a hydrocarbon.

Methane is an organic compound.
Organic compounds are studied by organic chemistry.
Methane is an organic compound.

Methane is studied by organic chemistry.

3. All tulips are flowers.
All flowers are plants.
All plants absorb carbon dioxide from the atmosphere and give off oxygen.
All plants which absorb carbon dioxide from the atmosphere and give off oxygen contain chlorophyll.

All tulips contain chlorophyll.

4. All that requires courage and heroism is an exploit.
The first flight by man into space demanded courage and heroism.

The first flight by man into space was an exploit.

Exploits are immortal.
The first flight by man into space was an exploit.

The first flight by man into space is immortal.

VI. Reconstruct the epihairems given below.

1. All fin-footed animals are aquatic mammals, since all fin-footed animals suckle their young.
   All walruses are fin-footed animals, since their extremities evolved into fins.

   All walruses are aquatic mammals.

2. All mammals are organisms, since all mammals breathe oxygen.
   All monkeys are mammals, since they suckle their young.

   All monkeys are organisms.

3. An honest and objective attitude towards oneself deserves respect, since it is a sign of high morals. Self-criticism expresses an honest and objective attitude of a person towards himself, since self-criticism is an open admittance of one's mistakes.

   Self-criticism deserves respect.

4. All plants are organisms, since all plants nourish themselves.
   All shrubs are plants, since all shrubs possess the property of photosynthesis.

   All shrubs are organisms.

5. Any crime is punished by law since it is a danger to society.
   Theft is a crime, since theft is the overt plundering of citizens' personal property.

   Theft is punished by law.

6. The hardening of the organism is beneficial, since the hardening of the organism helps to prevent illness.
   Morning gymnastics hardens the organism, since morning gymnastics strengthens the health.

   Morning gymnastics is beneficial.

VII. Define the type of inference, write down the formula and prove whether it is a law of logic.

1. If signs of rust have appeared on a metal, then corrosion has set in.
   Corrosion has not set in.

   No signs of rust have appeared on the metal.

2. If an official takes a bribe, he is committing a crime.
The said official does not take a bribe.

The said official is not committing a crime.

3. If this machine is an internal combustion engine, it is a thermal engine.
If this machine is a thermal engine, then the combustion of fuel in it occurs within the cylinder.

If this machine is an internal combustion engine, then the combustion of fuel in it occurs within the cylinder.

VIII. Determine the type of inference, write down the formula and prove whether it is a law of logic.

1. Phosphorus may be red or white.
This phosphorus is not red.

This phosphorus is white.

2. The zonal natural complexes of the East European plain are divided into tundra, forest-steppe, taiga, mixed forest, steppe, semi-desert and desert.
The said natural zone is taiga.

The said natural zone is not tundra, forest-steppe, mixed forest, steppe, semi-desert or desert.

3. In terms of the way sound is produced, stringed musical instruments are divided into bowed, plucked, hammered, keyboard-hammered, and keyboard-plucked.
This stringed musical instrument is not a bowed, plucked, hammered or keyboard-hammered instrument.

This musical instrument is a keyboard-plucked instrument.

IX. Determine the type of dilemma or trilemma and write down the formula.

1. If I go through the wood, I may be taken prisoner, and if I go through the field, I may be blown up by a mine.
I can go through the wood or through the field.

I may be taken prisoner or blown up by a mine.

2. The thoughts of a man who is shipwrecked in a storm near rocky shores:
If I swim towards the shore, I shall drown; if I swim towards the shore, I shall be smashed against the cliffs; if I swim towards the shore, I shall be eaten by the sharks. I shall not drown, and I shall not be smashed against the cliffs and I shall not be eaten by the sharks.

I shall not swim towards the shore.

3. If a patient has a respiratory illness, the patient is recommended to have cupping-glasses applied. If a patient has pneumonia, the patient is recommended to have cupping-glasses applied.
In this particular case, the patient either has a respiratory illness or pneumonia.

The patient is recommended to have cupping-glasses applied.

4. If the quantity of cholesterol in the blood plasma exceeds the norm, it is deposited on the walls of the vessels, the vessels lose their elasticity and blood pressure rises.
In the said case, either cholesterol is not deposited on the walls of the vessels, or the vessels do not lose their elasticity, or blood pressure is normal.

This means that the quantity of cholesterol in the blood plasma does not exceed the norm.

5. If Sasha is conscientious, he will do his homework; if he is a good friend, he will help his sick class-mate with his homework.
In the said case he did not do his homework and did not help his sick class-mate.

He is not conscientious or a poor friend.

6. If I take the metro from the theatre, I shall have to walk a long way home; if I take the trolleybus from the theatre, I shall have to walk along a dark street.
But I shall go from the theatre either by metro or by trolleybus.

I shall have to walk along a dark street or walk a long way home.

7. If I go to work by bus, I shall be late, and if I go by taxi, I shall spend a lot of money.
I may go to work by bus or by taxi.

I shall either be late for work or spend a lot of money.

8. If a patient has hypertension, then his arterial pressure rises, he gets a headache and his sight worsens.
It is known about the said patient that his arterial pressure has not risen, he has not got a headache or his sight has not become worse.

The said patient is not suffering from hypertension.

Chapter VI

THE LOGICAL FOUNDATIONS OF THE THEORY OF ARGUMENTATION

§ 1. The Concept of Proof

The cognition of individual objects and their properties occurs through forms of sensuous cognition (sensations and perceptions). We see that a building is not yet complete, feel the taste of a bitter medicine, etc. These truths are not subject to any special proof since they are obvious. However, we are called on in many cases, such as at a lecture, in an essay, a piece of academic work, a report, in a discourse, in court, in maintaining a thesis, and in many other instances, to prove, substantiate our judgements.

Provability is an important quality of correct thought.

In modern conditions, the theory of proof and disproof is a means of forming scientifically substantiated convictions. Those engaged in science are called upon to prove judgements of many varieties, for example, judgements on what existed before our era, on the period to which objects unearthed in archeological excavations belong, on the atmosphere of the planets in the solar system, on the stars and galaxies in the Universe, the mathematical theorems, the computer development prospects, long-term weather forecasts, the secrets of the World Ocean and space. All these judgements have to be substantiated.

Proof is the totality of logical approaches to substantiating the truthfulness of any judgement using other associated judgements which are true.

Proof is connected with conviction, but is not the same thing: proofs should be based on scientific data and socio-historical practice, whereas convictions may be based, for example, on religious beliefs, prejudices, people's ignorance, apparent provability founded on various kinds of sophism. Therefore, the act of convincing is rather less than that of rendering proof.

The structure of proof is: thesis, arguments, demonstration. The *thesis* is the judgement whose truth has to be proven. *Arguments* are true judgements used to prove the thesis. The *form of proof*, or *demonstration*, is the name given to the way a logical link is formed between the thesis and the arguments.

Let us give an example of proof.

Temperance and work are man's two true doctors: work arouses his appetite and temperance prevents it from being abused.

A distinction is made between various types of argument.

1. *Attested individual facts.* What is known as factual material belongs to this type of argument, i. e., statistics about population, the territory of a state, the fulfilment of a plan, the quantity of arms, evidence provided by witnesses, signatures on a document, scientific data, scientific facts. Facts play a very great role in substantiating propositions, scientific ones included.

In his *Letter to Youth*, the Soviet scientist Ivan Pavlov called on young scientists to "Study, compare and accumulate facts."

"No matter how perfect the wings of a bird, they could never raise it to a height without being supported by the air.

"Facts are the air for a scientist. Without them you will never be able to fly. Your theories will remain vain attempts.

"But in studying, experimenting and observing, try not to remain on the surface of facts. Do not turn into an archivist of facts. Try to penetrate the secret of their emergence. Persistently look for the laws which govern them."[1] The Soviet biologist and expert on selection Ivan Michurin said: "We cannot expect any favours from nature. It is our task to take them from it." By making tens of thousands of experiments and collecting scientific facts he created his own harmonious

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[1] I. P. Pavlov, *Selected Works*, Moscow, 1951, pp. 51-52 (in Russian).

scientific system for the development of new varieties of plants. To begin with, Michurin concentrated on studying the acclimatisation of Southern and West European varieties of fruits in the conditions offered by the central Russian belt. Michurin was able to create over 300 varieties of fruits and berries by means of hybridisation. This is an outstanding example of how a genuine scientist collects and processes a huge volume of factual scientific material.

In his article entitled "Statistics and Sociology", Lenin wrote the following about the role of facts in rendering proof: "Precise facts, indisputable facts ... are especially necessary if we want to form a proper understanding of this complicated, difficult and often deliberately confused question... Facts, if we take them in their *entirety*, in their *interconnection*, are not only stubborn things, but undoubtedly proof-bearing things. Minor facts, if taken out of their entirety, out of their interconnection, if they are arbitrarily selected and torn out of context, are merely things for juggling..."[1]

Lenin warned that it was inadmissible to select individual facts at random, to play at examples, since the selection of individual examples did not involve any effort: "And if it is to be a real foundation, we must take not individual facts, but the *sum total* of facts, without *a single* exception, relating to the question under discussion. Otherwise there will be the inevitable, and fully justified, suspicion that the facts are selected or compiled arbitrarily, that instead of historical phenomena being presented in objective interconnection and interdependence and treated as a whole, we are presenting a 'subjective' concoction to justify what might prove to be a dirty business. That does happen ... and more often than one might think."[2]

2. *Definitions as arguments of proof.* Definitions of concepts are formulated in every science. The rules and types of definitions were examined under "Concepts". We also gave numerous examples from various sciences, such as mathematics, chemistry, biology, geography, etc.

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[1] V. I. Lenin, "Statistics and Sociology", *Collected Works*, Vol. 23, 1977, p. 272.

[2] *Ibid.*, pp. 272–273.

3. *Axioms and postulates.* Apart from definitions, axioms are also used in mathematics, mechanics, theoretical physics, mathematical logic and other sciences. *Axioms* are judgements accepted as arguments without proof, since they have been confirmed by many centuries of human practice.

4. *Previously proven laws of science and theorems as arguments of proof.* Arguments of proof may take the form of previously proven laws of physics, chemistry, biology and other sciences as well as theorems in mathematics (both classical and constructive). The laws of materialist dialectics (the law of the unity and struggle of opposites, the law of the mutual transition of quantitative and qualitative changes and the law of the negation of negation) may also serve as arguments in the process of proof. Juridical laws are arguments in judicial proof.

In the course of proving any thesis, not one but several of the listed types of arguments may be used.

It should finally be stressed once again that the criterion of truth is practice. If practice has shown the truth of a judgement, then no further proof is necessary.

§ 2. Direct and Indirect Proof

In terms of form, proof may be direct or indirect. *Direct proof* proceeds from an examination of arguments to the proof of the thesis, i. e., the truth of the thesis is directly substantiated by the arguments. The pattern of this kind of proof is as follows: from the given arguments (a, b, c, ...) there necessarily follow (emerge) the true judgements (k, m, n, ...), and from the latter follows q, the thesis to be proved. This kind of proof is used in judicial practice, science, polemics, students' written work, by a lecturer in presenting material, etc.

Direct proof is widely employed in statistical reports, various kinds of documents and decrees.

Direct proof of the thesis "The people are the creators of history" may be rendered by the following arguments. First, the people are the creators of material assets; second, substantiation is provided of the enormous role played by the masses in politics and an explanation given of how in the current period nations are waging an active struggle for peace, democracy and socialism; third, illustration is provided of the major role played by the masses in creating intellectual culture.

In chemistry, direct proof of the combustibility of sugar may be provided in the form of a categorical syllogism:

All carbohydrates are combustible.
Sugar is a carbohydrate.
____
Sugar is combustible.

Indirect proof is a kind of proof in which the truth of the thesis is substantiated by proving the falsehood of the antithesis. It is employed when no arguments exist for direct proof. The antithesis may be expressed in one of two forms: 1) if we denote the thesis by the letter a, then its negation ($\bar{a}$) will be the antithesis, i.e., the judgement contradicting the thesis; 2) in the judgement $a \dot{\vee} b \dot{\vee} c$, the antithesis to thesis a is provided by judgement b and judgement c.

Depending on this distinction in the structure of the antithesis, indirect proofs are divided into two types – "negative proof" and disjunctive proof (by exclusion).

Negative proof is achieved by establishing the falsehood of the judgement which is contradictory to the thesis. This method is frequently employed in mathematics.

Let a be the thesis (or theorem) to be proved. Let us suppose the contrary, i.e., that a is false, and consequently *not-a* or ($\bar{a}$) is true. By assuming $\bar{a}$ we may derive consequences which conflict with reality or previously known theorems. We thus have $a \vee \bar{a}$, while $\bar{a}$ is false and its negation true, i.e., $\bar{\bar{a}}$, which, according to a law of two-valued classical logic ($\bar{\bar{a}} \rightarrow a$) gives a. This means that a is true, which is what we set out to prove.

It should be pointed out that the formula $\bar{\bar{a}} \rightarrow a$ cannot be derived in constructive logic, that is why it is not used in proofs in constructive mathematics and constructive logic. The law of the excluded middle is also "rejected" (not a derivable formula), so that indirect proofs are not applicable in this case.

There are numerous examples of negative proof in elementary mathematics. For example, this method is used to prove the theorem: "If two straight lines are perpendicular to one and the same plane, they are parallel". Proof of this theorem begins with the words "Let us assume the opposite, i. e., that the straight lines *AB* and *CD* are not parallel". Then they will intersect and form a triangle with two internal right angles, so that the sum of the triangle's three internal angles will be greater than 180°. But this contradicts the previously proven theorem that the sum of the internal angles of any triangle is equal to 180°. Consequently, the assumption that *AB* and *CD* are not parallel is false, from which it follows (according to the law of the excluded middle) as proven that the straight lines *AB* and *CD* are parallel.

Disjunctive proof (by exclusion). The antithesis is one of the elements of a disjunctive judgement, which should necessarily include all possible alternatives, for example:

The crime could have been committed by *A*, or *B*, or *C*.
It is proven that the crime was not committed by *A* or *B*.

The crime was committed by *C*.

The truth of the thesis is established by consecutively proving the falsehood of all elements of the disjunctive judgement except one.

Here we use the structure of the negative-positive modus of a disjunctive-categorical syllogism. The conclusion will be true if the disjunctive judgement provides for all possible instances (alternatives), i. e., if it is a closed (complete) disjunctive judgement.

$$\frac{a \vee b \vee c \vee d;\ \bar{a} \wedge \bar{b} \wedge \bar{c}}{d} \qquad (1)$$

As pointed out above, in this modus the conjunction "or" may by used as an exclusive disjunction ($\dot{\vee}$) or as an inclusive one (v), which means that it accords with two logical patterns (1 and 2).

$$\frac{a \dot{\vee} b \dot{\vee} c \dot{\vee} d;\ \bar{a} \wedge \bar{b} \wedge \bar{c}}{d} \qquad (2)$$

§ 3. The Concept of Disproof

Disproof is a logical operation geared to destroying a proof by establishing the falsehood or unsubstantiated nature of a previously advanced thesis.

A judgement which is to be disproved is called the *thesis of disproof*. Evidence that is used to disprove the thesis is called *arguments of disproof*.

There are three methods of disproof: 1) disproof of a thesis (direct and indirect); 2) criticism of arguments; 3) revelation of the invalidity of a demonstration.

1. Disproof of a thesis (direct and indirect)

A thesis may be disproved by the following three methods, the first being direct and the latter two indirect.

1. *Disproof by facts* is the most correct and successful method of disproof. We spoke in detail above about the role of selecting facts and the methodology of working with them; all these factors need to be considered in using facts contradicting a thesis as a method of disproof. Actual events, phenomena, statistics, witnesses' evidence and scientific data should be quoted which contradict the thesis, i. e., the judgement to be disproved. For example, in order to disprove the thesis "Organic life is possible on Venus" it is sufficient to quote the following facts: the temperature on the surface of Venus is 470 to 480 degrees Celsius and the pressure is 95 to 97 atmospheres. These facts indicate that life as we know it is not possible on Venus.

2. *Establishment of the falsehood (or contradiction) of consequences following from the thesis*. This method is used to prove that consequences follow from the thesis which contradict the truth. This method is called reduction to the absurd (*reductio ad absurdum*).

As already pointed out, in classical two-valued logic the method of reduction to the absurd is denoted by the formula $\bar{a} \underset{Df}{=} a \rightarrow F$, where F is a contradiction or falsehood.

In a more general form, the principle of reduction to the absurd is expressed by the formula: $(a \rightarrow b) \rightarrow ((a \rightarrow \bar{b}) \rightarrow \bar{a})$.

3. *Disproof of a thesis by proof of the antithesis.* In relation to the thesis to be disproved (judgement *a*), the contradictory judgement (i. e., *not-a*) is advanced, and *not-a* (the antithesis) is then proved. If the antithesis is true, then the thesis is false, and there is no third possibility.

For example, one may have to disprove the widespread thesis "All dogs bark" (judgement *A*, general and affirmative). For judgement *A* the contradictory judgement will be *O* (particular and negative): "Some dogs do not bark". In order to confirm the latter, it is sufficient to give just one example: "Pygmies'* dogs never bark". We have thus rendered proof of judgement *O*. By virtue of the law of the exluded middle, if *O* is true, then *A* is false. The thesis is thus disproven.

II. Criticism of arguments

The arguments presented by an opponent in support of his thesis are subject to criticism. The falsity or invalidity of these arguments is proven.

The falsity of the arguments does not necessarily imply the falsity of the thesis, and the thesis may remain true:

$$\frac{a \rightarrow b, \bar{a}}{\text{Probably } \bar{b}}.$$

It is not possible to draw a necessary conclusion by inferring from the negation of the antecedent to the negation of the consequent. But it is sometimes sufficient to show that the thesis has not been proved. It may happen that the thesis is true, but the person advancing it is unable to present true arguments to prove it.

* Pygmies – inhabitants of Central Africa, South-East Asia and Oceania, noted for their shortness.

III. Invalidation of the demonstration

This method of disproof involves revealing errors in the form of proof. The most widespread error is the presentation of arguments from which the truth of the thesis to be disproved does not follow. The proof may be improperly constructed if some rule of inference is violated or a "premature generalisation" (incorrect transition from the truth of judgement *I* to the truth of judgement *A* or from the truth of judgement *O* to the truth of judgement *E*) is made.

Having found fallacies in an argument, we invalidate the process of inference, but not the thesis itself. It is the job of the person who advanced it to prove the truth of the thesis.

All the above methods of disproving theses, arguments and the form of demonstration are frequently used in combination.

§ 4. The Rules of Provable Reasoning. Logical Errors in Proof and Disproof

Even if just one of the rules listed below is violated, errors may occur in connection with the thesis to be proved, the arguments or the actual form of demonstration.

I. Rules relating to the thesis

1. *The thesis should be logically defined, clear and accurate.* Sometimes, people are unable to accurately, clearly and unambiguously formulate a thesis in their speeches, written statements, academic articles, reports and lectures. At a meeting, several speakers may not formulate two or three theses precisely and then argue in their support before their audience. The audience would wonder why these people made speeches at all and what they were trying to prove.

2. *The thesis should remain identical*, i.e., it should be one and the same throughout the entire process of proof or disproof. The violation of this rule leads to the logical error of "thesis substitution".

Errors relating to the thesis to be proved

1. *Thesis substitution.* According to the rules of provable reasoning, the thesis should be clearly formulated and remain one and the same throughout the entire process of proof or disproof. If these rules are violated, an error occurs, known as thesis substitution. This happens when one thesis is deliberately or accidentally substituted for another and a start then made on proving or disproving this new thesis. This occurs frequently in an argument or a discussion, when the opponent's thesis is either oversimplified or its content extended and then subjected to criticism. The person being criticised objects that his opponent is ascribing something to him which he never actually said. This situation is quite common and encountered in maintaining dissertations, in discussing published academic works and at various kinds of meetings and assemblies as well as in the editing of scientific and literary articles. The law of identity is violated, since an attempt is made to identify non-identical theses, which leads to a logical error.

2. *Reference to personal qualities.* This error is the substitution of the proof of the actual thesis with references to the personal qualities of the individual who advanced it. For example, instead of proving the value and the innovatory nature of a dissertation, it is said that its author is a deserving person who worked hard on the dissertation, etc.

In scientific works, a concrete analysis of the material concerned, the study of up-to-date scientific data and the results of practice are replaced by quotations from outstanding scientists and prominent figures, with no further evidence being given, since it is assumed that mere reference to authority is sufficient in itself. Moreover, quotations may be taken out of context and sometimes given a subjective interpretation. Reference to *personal qualities* is often simply a sophistic device rather than an accidental error.

A variety of the above device is an appeal to people's feelings so that they believe the truth of an advanced thesis even though it cannot be proved.

3. *Overstatement and understatement of a thesis.* There

are two types of this error: (a) he who proves too much proves nothing at all; (b) he who proves too little proves nothing at all.

In the first case, an error occurs when an attempt is made to prove a stronger thesis instead of the original true one, and the second thesis may turn out to be false. If *b* follows from *a*, but *a* does not follow from *b*, then thesis *a* is stronger than thesis *b*. For example, if, instead of proving that this man was not the one to start the fight an attempt is made to prove that he did not take part in the fight, then nothing at all will be proven if he did take part in it and someone saw him.

The error "he who proves too little proves nothing at all" is made when a weaker thesis *b* is proven instead of a stronger one *a*. For example, if we try to prove that this animal is a zebra and succeed in proving that the animal is striped, then we prove nothing at all, since a tiger is also striped.

II. Rules relating to arguments

1. The arguments advanced in support of a thesis must be true and non-contradictory.
2. Arguments in support of a thesis are to be conclusive.
3. Arguments are to be presented in the form of judgements whose truth has been demonstrated independently of the thesis.

Errors in the basis (arguments) of proof

1. *Falsity of the basis (Basic errors)*. Not true, but false, judgements are taken as arguments and they are passed off as true, or an attempt is made to do this. The error may be not deliberate. For example, Ptolemy's geocentric system was constructed on the basis of a false assumption that the sun rotates around the earth. The error may also be deliberate (sophism), made with the intention of confusing or misguiding other people (for example, the submission of false evidence by witnesses or the accused in the course of investigation, the incorrect identification of things and people, etc.).

2. *Anticipation of the basis*. This error is made when

the thesis is supported by unproven arguments, which do not prove it but merely anticipate it.

3. *Vicious circle.* This error is made when a thesis is supported by arguments which are supported by this same thesis. It is a variety of the error of "using an unproven argument". An example of this is the error made in the reasoning of John Weston, a functionary in the British working-class novement. Referring to this error, Marx writes: "Thus we begin by saying that the value of labour determines the value of commodities, and we wind up by saying that the value of commodities determines the value of labour. Thus we move to and fro in the most vicious circle, and arrive at no conclusion at all."[1]

III. **Rules referring to the form of supporting a thesis (demonstration) and errors in the form of proof**

A thesis is to be a conclusion logically following from the evidence in conformity with the general rules of inference or obtained by the rules of indirect proof.

Errors in the form of proof

1. *Imaginary consequence.* If a thesis does not follow from the arguments presented in its support, then we have an error of "does not follow". Sometimes, instead of constructing proper proof, the arguments are linked with the thesis by connectives: "therefore", "thus", "in this way", "as a result, we have", etc. It is assumed that a logical connection is thus established between the arguments and the thesis. This logical error is often made by people who are unfamiliar with the rules of logic and rely on their common sense and intuition. This kind of argument presents ostensible verbal proof.

As an example of the logical error of imaginary consequence, B. A. Vorontsov-Velyaminov in his astronomy textbook referred to the widespread opinion that the spherical shape of the earth is proven by the follow-

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[1] Karl Marx, "Value, Price and Profit", in: Karl Marx and Frederick Engels, *Collected Works*, Vol. 20, 1985, p. 120.

ing observations: (1) as a ship approaches the shore from behind the horizon, its masts first become visible and then the hull of the ship; (2) voyage round the world, etc. But from these arguments it does not follow that the earth has the shape of a sphere (or, more precisely, a geoid), but merely that the earth has a curved surface and is a continuous form. In order to prove that the earth is spherical, B. A. Vorontsov-Velyaminov provides the following observations: (1) at any point on Earth, the horizon is circular and the length of the horizon is everywhere the same; (b) during a lunar eclipse, the shadow of the earth which falls on the moon is circular, which is only possible if the earth is spherical.

2. *Arguing from what has been stated conditionally to what has been stated unconditionally*. An argument which holds true only under special conditions (time, relation or degree) is not to be presented as a conclusive one and true for all instances. For example, while coffee is beneficial in small quantities (to raise arterial pressure, for instance), it is harmful in large quantities. Similarly, arsenic is poisonous, but in small doses it is added to some medicines. Doctors should select medicines for patients on an individual basis. Educational science demands an individual approach to pupils; ethics defines the standards of human behaviour, but they may vary in different conditions (for example, frankness is a positive human quality, but the disclosure of a military secret is a crime).

Violation of the rules of inference
(deductive, inductive and by analogy)

1. *Errors in deductive inference*. For example, it is inadmissible to construct an inference from the assertion of the consequent to the assertion of the antecedent. From the premises: "If a number ends in 0, it is divisible by 5" and "This number is divisible by 5", it does not follow that "This number ends in 0". Errors in deductive inference were discussed in detail above.

2. *Errors in inductive inference*. One of these errors is "premature generalisation", e. g. the assertion that "all witnesses give unobjective evidence". Another error occurs when "thereafter" is interpreted as "therefore"

(for example, "the thing disappeared after this man came into the building which means that he took it away"). All superstitions are based on this logical error.

3. *Errors in inference by analogy.* A partial example of this may be provided by the ritual dances of African pygmies preceding an elephant hunt. Apart from other mystic notions, this ritual betrays an obvious analogy.

> An elephant hunt demands special preparations. Evil spirits must be appeased and the moral support of all villagers obtained. On the eve of the hunt, the village is the scene of a veritable show in which the hunters, having made an elephant's dummy, show their fellow tribesmen how they are going to hunt. The "actors" begin by moving cautiously, listening attentively and looking ahead. They maintain contact with each other by signs... Then the drums enter the game. They call out loudly, indicating that the hunters have found the track.
>
> Suddenly the drums thunder "Boom". The leader draws himself up, waves at his companions and looks with mixed fear and triumph at the elephant's dummy which at this moment seems like a veritable living giant to all those present. The hunters stand still for several seconds, their eyes fixed on the elephant. Then they withdraw seven or eight paces and start to discuss the plan of attack. The leader is to be the first to strike the elephant with a spear. He sneaks up to the elephant from behind, but suddenly his eyes grow wide with fear, as if the elephant is about to turn around, and he makes for the forest as fast as his feet will carry him. Three times the leader steals up to the elephant, and three times he runs away. Then the hunters throw themselves at the elephant, furiously stick their spears into the dummy and turn it over. The hunters perform their victory dance around the vanquished dummy. Five minutes later, the entire audience is joining in the dance to the accompaniment of drums.

§ 5. The Concept of Sophisms and Logical Paradoxes

An unintentional mistake in human thought is called a *paralogism.* An intentional mistake (as already pointed out on repeated occasions) made with a view to confusing an opponent and passing off a false judgement for a true one, is called a *sophism.* Sophists is the name given to reasoners who use various verbal subtleties to present falsehood as truth. The sophisms of the idealists were exposed by Lenin. Many examples of this can be found in his work *Materialism and Empirio-Criticism.*

In analysing the concepts of various representatives of empirio-criticism, Lenin came to the conclusion that

"No evasions, no sophisms (a multitude of which we shall yet encounter) can remove the clear and indisputable fact that Ernst Mach's doctrine that things are complexes of sensations is subjective idealism..."[1]

In exposing the political and philosophical sophisms of his opponents, Lenin compared them with mathematical sophisms. He wrote that they were "as like as two peas to those arguments which mathematicians call mathematical sophistries, and which prove–quite logically, at first glance–that twice two are five, that the part is greater than the whole, and so on".[2]

Mathematical sophisms are collected in a whole number of books. The following are examples of such sophisms: (1) "5 = 6"; (2) "2 × 2 = 5"; (3) "2 = 3"; (4) "All numbers are equal to each other"; (5) "Any number is equal to one half of itself"; (6) "A negative number is equal to a positive one"; (7) "Any number is equal to 0"; (8) "Two perpendiculars may be drawn from a point to a straight line"; (9) "A right angle is equal to an obtuse angle"; (10) "Any circle has two centres"; (11) "The lengths of all circles are equal", and many others. For example, 2 × 2 = 5. We are required to find the error in the following arguments. We have a numerical identity: $4:4 = 5:5$. Let us put a common multiplier out of the brackets in each half of this identity. We obtain $4\,(1:1) = 5\,(1:1)$. The numbers in brackets are the same. Therefore $4 = 5$, or $2 \times 2 = 5$.

$5 = 1$. Seeking to prove that $5 = 1$, we shall reason as follows. Let us separately subtract from 5 and 1 one and the same number 3. We will obtain the numbers 2 and -2. If these numbers are raised to their square, we obtain 4 in both cases. This means that the original numbers 5 and 1 must be equal. Where is the error?

The concept of logical paradoxes

A *paradox* is a piece of reasoning which proves both the truth and falsity of some judgement; in other words,

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[1] V.I. Lenin, "Materialism and Empirio-Criticism", *Collected Works*, Vol. 14, p. 42.

[2] V.I. Lenin, "The Position of the Bund in the Party", *Collected Works*, Vol. 7, 1974, p. 94.

it proves the judgement as well as its negation. Paradoxes were familiar as far back as ancient times. Examples of paradoxes are: "heap", "bald", "catalogue of all normal catalogues", "mayor of a town", "general and barber", etc.

Paradox "heap". One grain of sand does not make the difference between a heap and a non-heap. Let us have two heaps (say, of sand). We may start and take one grain away each time, but the heap will remain a heap. Let us continue this process. If 100 grains represent a heap, then so do 99... 10... 2 grains and 1 grain.... The paradox is thus that continual quantitative changes (reduction by one grain) do not lead to qualitative changes.

The paradox "bald" is similar to the paradox "heap", i. e., the difference between bald and not bald is not to be found in one hair.

Paradoxes in the theory of sets

Bertrand Russell detected a paradox in Gottlob Frege's all normall sets (a normal set is one which does not contain itself as an element).

Examples of such paradoxes are "catalogue of all normal catalogues", "mayor of a town", "general and barber", etc.

The paradox called "mayor of a town" consists in the following: every mayor of a town lives in that town or outside it. An order was issued on the establishment of a special town only for mayors living outside their own towns. Where should the mayor of the special town live? If he wishes to live in his town, he cannot do this, since it is a town only for mayors living outside their own towns; if he does not wish to live in his own town, then, like all mayors not living in their own towns, he must live in the specially allocated town, i. e., his own. He can thus neither live in his own town nor outside it.

The *paradox "general and barber"* consists in the following: every soldier may shave himself or have another soldier shave him. The general issues an order about the appointment of a soldier-barber to shave only those soldiers who do not shave themselves. Who should shave this special soldier-barber? If he wishes to do it himself, he cannot, since he can only shave those soldiers who do not shave themselves; if he does not shave himself, then, like all other soldiers who do not shave themselves, he should be shaven by the special soldier-barber, i.e., himself. He can thus neither shave himself nor not shave himself.

This paradox is similar to the paradox "mayor of a town".

Let us examine Russell's paradox of normal sets in the form of the paradox of the catalogue of all normal catalogues.

The paradox is as follows: catalogues are divided into two types: (1) those which do not include themselves in the list of catalogues (normal) and (2) those which do include themselves in the list of catalogues (abnormal sets).

A librarian is given the job of drawing up a normal catalogue of all normal, and only normal, catalogues. In compiling the catalogue, should he mention the one which he is compiling? If he does mention it, then the catalogue he has compiled will not be a normal one, i.e., he does not have the right to mention it. If he does not mention it, then one of the normal catalogues, i.e., the one he has compiled, will be omitted, although he is required to mention all normal catalogues. He can thus neither mention nor not mention the catalogue he has himself compiled.

What is the solution? This example shows how such paradoxes may be resolved. It is only natural to state that the concept "normal catalogue" has no fixed extension, that it has not yet been established which catalogues should be included: in which library they are to be found, for example, and which time they refer to. If someone is given the job of compiling a catalogue of all normal catalogues as of May 10, 1988, then the extension of the concept will be fixed, and the librarian will have no need to mention his own catalogue when compiling it. But if he is given a similar job after the previous catalogue has been compiled, then he will have to mention this catalogue as well. This is the solution to the paradox.

Logic thus embraces the category of time and the category of change, which entails the need to consider changing extensions of concepts. The consideration of an extension in the process of change is, however, an element of dialectical logic. The interpretation of paradoxes in mathematical logic and of the theory of sets associated with the violation of the demands of dialectical logic was a task undertaken by the Soviet mathematician and logician S. A. Yanovskaya.

There are also other methods of avoiding this kind of contradiction.

§ 6. Proof and Debate

The role of proof in scientific cognition and disputes boils down to the selection of effective antecedent (arguments) and the demonstration of the fact that the thesis being proved follows from it with logical necessity.

The rules for conducting a debate may be illustrated using the example of a youth dispute. The debate facilitates the evolution of an active ideological and moral stance. The argument makes it possible to examine and analyse problem situations, and develop an ability to effectively support one's knowledge and convictions.

Debates may be planned in advance or crop up spontaneously (in the course of a trip, following a film, etc.). In the first instance, it is possible to read the appropriate literature and be ready in advance. The advantage of the second instance is the emotional attitude of the participants to the subject of discussion. It is extremely important to choose the subject of the debate, since it must sound controversial and problematic. For example, one could choose such subjects as "Your ideals"; "How can one acquire an ability to independently develop one's knowledge and find one's bearings in a huge flow of scientific and political information?"; "A question to oneself: 'What have I done today that is of benefit to others?' "; "Is it only you alone who has the right to set requirements of yourself?", etc.

In the course of an argument, it is necessary to ask three or four questions of a kind to which there are no clear-cut answers. For example, these questions suggest themselves for a debate on the subject "Your principles – do you stand by them?".

1. What does it mean to be principled?
2. What do you think is more useful in life: prudence or straightforward frankness?
3. Principles, tact, delicacy – how can these be balanced?
4. Convictions – how do you think they should manifest themselves?

It is advisable to spend one or two months preparing for such a debate. Students' opinions should be sought by using questionnaires, their answers studied and generalised. The recommended literature is to be studied in advance.

For example, in the preparations for a debate, the following "rules of debate" were compiled (by the students themselves).

Before arguing, think of the most important thing you wish to prove;

If you have entered into a debate, be sure to make a contribution and prove your point of view;

Speak simply and clearly, logically and consistently;

Speak only of that which worries you, of which you are convinced, do not assert something which you yourself are not quite clear about;

Argue honestly; do not distort the thoughts of someone with whom you do not agree;

Do not repeat what has already been said before you;

Do not wave your arms about, do not raise your voice: the best proof is accurate facts and iron logic;

Respect those who argue with you: try not to offend your opponent; acting in this way you show that you are not only strong in arguments but also well-bred.

These rules were displayed on a brightly coloured poster which announced the debate and was displayed one or two days before it took place.

The Soviet educationalist V. A. Sukhomlinsky referred to the great power of words and to the need to use them with tact, since things said in haste may cause a great deal of harm. He warned that unreasoned, cold and indifferent wording may offend, hurt, embitter and shock the opponent. For some to speak is to offend: they are sharp and caustic, their speech is a mixture of bile and absinth; sneers, jibes and insults flow from their lips like saliva. And, conversely, great is the role of kind words. It happens that one or two are enough to make a person happy.

Debates demand considerable preparatory work. During the discourse, the chairman should not interrupt the participants. The summing-up must not amount to moralising nor to an attempt to judge the opponents: what should be stressed are the collective findings and the conclusions which have been reached independently, and questions should be posed for further discussion.

§ 7. The Logical Structure of Questions and Answers

Questions are formulated not only to resolve new problems and tasks in science and practice, but also to assimilate already obtained knowledge, in educational work. In cognition, questions play an especially great role, since the entire cognition of the world begins with questions and the formulation of problems. The problems requiring cognition, including those confronting the various sciences, are posed by practice, since it represents the basis of cognition. At the present time, socio-historical practice has faced man with such problems as achieving a controlled thermonuclear fusion, fighting for peace and the prevention of a thermonuclear catastrophe, working out methods of curing oncological diseases, solving the food problem and many others. There is no area of occupational employment where questions do not crop up.

The terms "problem", "question" and "problematic situation" are not identical concepts, although they are connected with each other. A problem is a question which cannot be answered on the basis of the information (knowledge) available at a given time. One of the ways of solving a problem is by using a hypothesis. A question is a form of expressing a problem. But a question is also asked with the aim of obtaining some information which a person already has in his possession, with the aim of finding out his personal opinion on the issue at hand or of instructing. Questions play a great role in the process of sociological surveys carried out in the form of interviews, questionnaires, mass or sample polls. As an increasing number of intellectual functions are assumed by computers, an ability to formulate a request clearly and correctly, to feed it into a computer, will facilitate the rapid search for required information, digital material, etc. The correct and unambiguous formulation of questions plays a major role in judicial and investigative practice.

A considerable role in developing the students' logical reasoning accords to the teacher's ability to pose questions and obtain correct answers to them, answers which facilitate the intellectual development of the students and give an impetus to an independent brain-work. Questions may be different in complexity, vary in the demands they make on students' knowledge, differ in form and take account of the students' age.

Small children are presented with pictures which help them to comprehend the reality surrounding them. For example, pictures with the general heading "Why does it happen like that?". A willow shoot planted in moist ground sprouts and grows into a tree, but an oak shoot withers. Why? The second series of pictures is "Why is that done?" For example, why is a hole made in the thick ice of a pond in winter? In very hot weather, the dry soil around vegetables is spread with fine humus. Why? The pictures in the third series are headed "What is wrong here?". They contain deliberate mistakes: for example, tomatoes ripening in the thick shadow of an oak tree; the shadow of poplars falls on the side from which the sun is shining; beehives are erected on a part of a field sown with wheat. On one of the pictures in the fourth series – "Where does this happen?" – we find an aeroplane landing on a small area surrounded by hummocky icefields. The section "Why is this done?" has pictures with the captions "Why is coal soaked in water before burning?"; "Why are the metallic parts of machines smeared with grease for winter?" Then there is the series, "How can we know?" Will the apple blossom come out in spring – – how can we know this in winter? And so on, and so forth.

Children's thought is directed towards seeking answers to questions. Arousing such a need in a child means getting him used to mental work. The most difficult thing and the most faithful indicator of a teacher's mastery is an ability to ask questions.

A question is formulated in the form of an interrogative sentence which does not express any judgement and is consequently neither true nor false. For example: "When was Voltaire born?", "Has any artificial Mars satellite been launched?"; "Are all volcanoes mountains?", etc.

Every question includes, first, basic information about the world (about artificial satellites, for example), this being called the basis or the prerequisite of a question, and, second, an indication of its insufficiency and the need for supplementary and more profound knowledge.

A question is a logical form including basic information and at the same time indicating its insufficiency with a view to obtaining new information in the form of an answer.

Types of questions

We normally distinguish between two kinds (types) of questions:

Type I questions are specifying (definite, direct or "is it.." questions).

For example, "Is it true that I. S. Vasilyev won the marathon skiing event?", "Is it necessary to pass an entrance exam in a foreign language for the history faculty of Moscow University?"; "Is it true that Delhi has more inhabitants than Bombay?", etc.

Specifying questions may be simple or complex. Simple questions may in turn be conditional or unconditional.

"Is it true that cosmonauts have been on the moon?" is a simple unconditional question.

"Is it true that he will receive a higher grant if he obtains top marks in all the exams?" is a simple conditional question.

Complex questions (just like complex judgements) may be conjunctive or disjunctive and contain exclusive or inclusive disjunction. For example:

1. "Does the USSR occupy first place in the world in the production of iron ore and steel, the extraction of oil and coal?"
2. "Would you like coffee or tea?"
3. "Are you going to the cinema or not?"

Questions of the type: "Shall we go on an excursion if the weather is good?" and "Will *Dynamo* get to the final if they beat *Spartak* in this match?" are not complex questions since they cannot be divided into two independent simple questions. They are examples of simple conditional questions.

Type II questions are supplementing (indirect, *W*-questions). They include in their composition the interrogative words "Where", "When", "Who", "What", "Why", "Which", etc. These questions may also be either simple or complex. For example, "Which city is the capital of Portugal?", "What does the word 'Philistine' mean?" are simple questions geared to obtaining knowledge which is lacking in order to supplement insufficient information.

Complex supplementing questions are those which can be split up into two or more supplementing questions, like: "How do the perimeter and area of an equilateral triangle change when the length of its sides is doubled?", "Who is the author of the novels *Quiet Flows the Don* and *Virgin Soil Upturned*?"

Prerequisites of questions

The prerequisite, or basis, of a question is the original knowledge contained in the question whose incompleteness or lack of definition has to be removed. The operators of the question indicate this incompleteness or lack of definition, i.e., the interrogative words "Who", "What", "When", "Why", etc.

Questions may be logically correct (correctly posed), i.e., those whose prerequisites (bases) are true judgements, or logically incorrect (incorrectly posed), whose prerequisites are either false or ambiguous (in terms of sense) judgements. If the inquirer does not know that the basis of his question is false, the question is said to be incorrect. If the inquirer is aware that the basis is false, but seeks to provoke or confuse his opponent, then the question is provocative and its posing is a sophistic device.

For example, the question "When was Roald Amundsen the first to reach the North Pole?" is posed incorrectly, since the person asking it may not know that Amundsen was the first to reach the *South Pole* in 1911.

Examples of provocative questions are the following: "In what way can unemployment be abolished under capitalist conditions?", "What is the shape of a flying saucer?", and "How do you build a perpetual motion machine?".

The prerequisites of the questions are deliberately false, so these questions are posed incorrectly and the very act of posing them is a sophistic ruse.

Rules for posing simple and complex questions

1. Questions should be posed correctly. Provocative and ambiguous questions are inadmissible.
2. Specifying questions should provide for alternative answers ("yes" or "no"). For example, "Did it rain yesterday in the centre of Moscow?", "Does this man plead guilty to the charge levelled against him?".
3. A question should be formulated in a concise and clear way. Long, confused and vague questions are more difficult to understand and answer.
4. A question should be simple. If a question is complex, it is better to split it up into several simple ones. For example, "Were Czechoslovakia and Mongolia members of the CMEA in 1960?" This complex question should be split into two simple ones, since the answers will be different ("yes" and "no"). In 1960, Czechoslovakia was a member of the CMEA and Mongolia was not, since it did not join until 1963.
5. In complex disjunctive questions, it is essential to list all alternatives. For example, "Was the said piece of scientific work by S. M. Popov awarded the first, second or third prize?" Here there is no fourth alternative – that the work was not awarded any prize at all.
6. The need to distinguish between a conventional question and a rhetorical one (for example, "Is there anyone who does not love Pushkin?"). Rhetorical questions are judgements, since they contain an affirmation or a negation, while conventional questions do not constitute judgements.

The logical structure and types of answers

1. Answers to simple questions. The answer to a simple question of the first type (specifying, definite, direct or "is it" question) is "yes" or "no". For example, "Is Alexander Dumas Sen. the author of the novel *Twenty Years After*"? The answer is "Yes".

The answer to a simple question of the second type (supplementing, indirect, *W*-question) demands the introduction of precise, exhaustive information (on the time, place, causes, results of an event, natural phenomenon and other factors).

2. Answers to complex questions. An answer to a complex conjunctive question calls for answers to the simple questions which make up the complex one. For example: "Is it true that the tincture of ginseng is used as a tonic in cases of low blood pressure, fatigue and nervous exhaustion?" (answer: "yes", "yes", "yes").

In replying to a complex disjunctive question, it is sufficient to give an answer to one or several of the simple questions (one alternative) which goes to compose it. For example, the question "Do you prefer to travel or take it easy by the river in summer?" may be answered by "I prefer to take it easy by the river in summer."

At the beginning of the section we referred to the major role played by questions in cognition. Let us dwell on the equally important role of questions in teaching, since the assimilation of the material and the achievement of pupils greatly depends on questions being posed in a correct manner.

In the process of teaching, one may use the following classification of questions. The first type of questions are direct, which may be answered by "yes" or "no". The second are direct ones which may not be answered in such a clear-cut manner. The third are questions in which it is asked which of two (or more) judgements is true. In the latter case, the complex question must be split up into several simple ones.

In answering a question, pupils should reveal the prerequisites of the question and establish whether they are true or false. If the prerequisites are false, the question should be rejected as incorrect, i. e., incorrectly posed, for example: "Are all geysers volcanoes?".

Correct questions should give rise to intensive thought activity if they include the optimal quantity of indefiniteness. If a question contains too much indefiniteness, it greatly puzzles the pupil. "Easy" questions with little indefiniteness enable pupils to answer using the words in the textbook and do not call for any investigation or examination of individual instances.

For example, instead of the question: "How many circles can be drawn through three points not lying on one straight line?" (the easy answer is "one"), it is better to ask "Is there any circle passing through three points?", since there is no ready answer to this in the textbook and the pupils will have to examine various instances of the location of three points (on one straight line or not on one straight line).

One should avoid asking vague questions, for example: "What may be said about triangle *ABC*?", "What are the properties of a trapezium?", "What properties do not belong to a cube?".

Problems on proof

I. Find the thesis, the arguments and indicate the method of proof.

1. "This valley is a wonderful place indeed! You are surrounded on all sides by inaccessible mountains, reddish cliffs hung with green ivy and crowned with plane-tree groves, yellow precipices furrowed with ravines, and up on high the golden fringe of the snows, down below the Aragva embracing another nameless river, loudly rushing from a gorge filled with dark gloom, runs like a silver line and glitters like the scales of a snake."[1]

2. "Passions lead us astray, since they focus our entire attention on one aspect of the object being considered and do not permit us to investigate it from all aspects."[2]

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[1] M. Yu. Lermontov, "A Hero of Our Times", *Collected Works* in four volumes, Moscow-Leningrad, 1953, Vol. 1, p. 277 (in Russian).

[2] Helvetius, *De l'Esprit*, Vol. 1, London, 1777, p. 18.

3. "Death is nothing to man, since when we exist death is not yet present and, when death is present, we do not exist."

4. "Only a fool is intrusive: a wise man feels immediately whether his company is welcomed or annoying and withdraws a second before it becomes clear that he is not wanted."

II. Find the logical and mathematical error in the following reasoning (solution of the problem).

Prove that $2 \times 2 = 5$.

We take the equation $16 - 36 = 25 - 45$. Then the value $20^1/_4$ is added to each part of the equation to obtain: $16 - 36 + 20^1/_4 = 25 - 45 + 20^1/_4$. We then carry out the following transformations:

$$4^2 - 2 \times 4 \times {}^9/_2 + ({}^9/_2)^2 =$$
$$= 5^2 - 2 \times 5 \times {}^9/_2 + ({}^9/_2)^2,$$
$$(4 - {}^9/_2)^2 = (5 - {}^9/_2)^2.$$

From this we may conclude that $4 - {}^9/_2 = 5 - {}^9/_2$, $4 = 5$, $2 \times 2 = 5$. Where is the mistake?

III. Where are the logical fallacies made in the following sophisms?

1. You have everything which you have not lost.
   You have not lost horns.
   ___
   You have horns.

2. In ancient times, the sophism "Euathlus" was well known. The ancient Greek sophist Protagoras gave lessons to Euathlus. They agreed that after Euathlus had won his first court case the pupil would pay his teacher for the instruction received. But Euathlus did not conduct any court case and thus did not pay his teacher for the instruction. Protagoras said that he would take Euathlus to court and Euathlus would then have to pay him: if the judges ordered that he must pay, then he would be forced to pay on the decision of the court, and if the judges did not order him to pay, then Euathlus would have to pay for his instruction under the terms of the agreement, since he would have won his first court case.

To this Euathlus replied that he would pay in neither case, since if the judges ordered him to pay, he would have lost his first case and would therefore not be bound to pay under the terms of their agreement and, if the judges did not order him to pay, he would not pay in accordance with the court's decision. Why did this sophism arise?

IV. On what logical laws does the proof (i. e., solution) rely in the problems below?

A, *B* and *C* were brought before the court accused of burglary.

The following was established:

(1) If *A* is not guilty or *B* is guilty, then *C* is guilty.

(2) If *A* is not guilty, then *C* is not guilty.

Is it possible on the basis of this information to establish the guilt of each of the three accused?

Solution

It is possible, and very easily at that. By virtue of statement (1), if *A* is not guilty, then *C* is guilty (due to the fact that, if *A* is not guilty, then the disjunction "either *A* is not guilty or *B* is guilty" is true). By virtue of statement (2), if *A* is not guilty, then *C* is not guilty. Thus, if *A* is not guilty, then *C* is both guilty and not guilty at the same time. This means that *A* must be guilty.

Chapter VII
HYPOTHESIS

§ 1. Hypothesis as a Form of the Development of Knowledge

In science, investigative practice and everyday thought we proceed from ignorance to knowledge, from incomplete to more complete knowledge; we are called upon to advance and then substantiate various assumptions to explain phenomena and their connection with other phenomena. We put forward hypotheses which, if they are confirmed, may become scientific theories or individual true judgements. They may conversely be refuted and turn out to be false judgements.

Hypothesis is a scientifically substantiated assumption about the causes of, or law-governed links between, any phenomena and events in nature, society or thought. Engels attached great importance to hypotheses in the process of cognition and called hypothesis a form of the development of natural science.

Scientifically substantiated assumptions (hypotheses) should be distinguished from the fruits of groundless imagination in science. In a letter addressed to young scientists, Pavlov warned against advancing empty hypotheses. He wrote: "Never try to cover up shortcomings in your knowledge even with the boldest guesses and hypotheses. No matter how much this soap bubble beguiles the eye, it will inevitably burst and you will be left with nothing but shame."[1]

There are also false hypotheses, such as that which existed before Copernicus about the immobile nature of the Earth. A new heliocentric system was devised by

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[1] I. P. Pavlov, "A Letter to Youth", *Selected Works*, Moscow, 1951, p. 50 (in Russian).

Nicolaus Copernicus (1473–1543) in a fundamental work called *On the Revolutions of the Heavenly Spheres.* Seventy-three years after its publication, this book was put on the prohibited list by the Vatican, where it stayed until 1822. In examining the arguments advanced by the advocates of the geocentric system, which was dominant at that time, Copernicus wrote: "And so, in the process of demonstration which they call 'method', they are found either to have omitted something necessary or to have admitted something foreign which by no means pertains to the matter; and they would by no means have been in this fix, if they had followed sure principles. For if the hypotheses they assumed were not false, everything which followed from the hypotheses would have been verified without fail."[1]

Hypothesis is a form of developing the natural, social and engineering sciences: from the angle of logical structure it does not boil down to any single form of thought – concept, judgement or inference – but includes all these forms.

Types of hypotheses

Depending on their level of generality, scientific hypotheses may be divided into general, particular and individual.

General hypothesis is a scientifically substantiated assumption about the causes, laws and rules governing natural and social phenomena, and also the laws of human psychic activity. General hypotheses are advanced with a view to explaining the entire class of the phenomena described and revealing the necessary nature of their interconnections at any time and place. An example of a general hypothesis is Democritus' hypothesis about the atomic structure of substance, which subsequently became a scientific theory; another example is provided by the hypotheses about the organic and inorganic origin of oil, etc. If confirmed, a general hypothesis becomes a scientific theory.

[1] Nicolaus Copernicus, "On the Revolutions of the Heavenly Spheres", in: *Great Books of the Western World*, William Benton, Chicago, 1952, pp. 507–508.

Particular hypothesis is a scientifically substantiated assumption about the causes, origin and regularities of a part of objects singled out from a class of objects belonging to nature, the life of society or human psychic activity.

Particular hypotheses are devised to reveal the reasons for the emergence of regularities in a subset of the given set.

Here, for example, are three contemporary particular hypotheses: A serious philosophical and general biological problem is the origin of viruses. Paleontologists working on the origin of plants and animals have at their disposal mineral remains which enable them to follow the main stages and branches of evolution in general terms. As for fossil viruses, no scholar met them even in a pipe-dream. For the moment, research amounts to working out hypotheses. According to one of them, viruses originate from normal cell components which refused to obey the regulation mechanisms. According to another hypothesis, viruses are the descendants of bacteria which have adopted a parasitic mode of life within the cell. In the process of evolution their ancestors in the form of bacteria lost the ability to carry out their metabolic functions independently and were deprived of their cell membrane. The hypothesis that viruses originate from primary precellular forms of life looks more plausible. However, none of the assumptions has yet been conclusively proved. The danger of viruses is that, according to many experts, viruses currently reduce the world harvest by 70 to 80 per cent.

There also exist several particular hypotheses about the causes of malignant tumours, including the hypothesis about oncogenic RNAs containing viruses.

Among the many problems connected with preparations for prolonged space flights, the most serious and least solved is that of how man can coexist with viruses in the closed atmosphere of spaceships. Research into virology thus represents an extremely important aspect of work in the field of biology, and the transformation of hypotheses into scientifically substantiated theories will be of great scientific and practical importance.

We refer to hypotheses in virology as particular, and not general, because they are advanced with a view to revealing the laws governing individual organisms, only a part of the total, namely viruses – and sometimes not even all viruses, but their individual varieties.

Individual hypothesis is a scientifically substantiated assumption about the causes, origins and regularities of individual facts, concrete events or phenomena. A doctor constructs individual hypotheses in the course of treating an individual patient, selecting drugs and their dosage individually.

In order to prove a general, particular or individual hypothesis, people construct *working hypotheses*, i. e., assumptions usually advanced at the beginning of research into a phenomenon and not yet seeking to clarify its causes or regularities. A working hypothesis allows the researcher to systematise or group the obtained results and to formulate a corresponding preliminary description of the phenomenon being studied. The work carried out by Academician Pavlov is a vivid illustration of the means and aims of constructing a working hypothesis. One of his pupils and colleagues, Academician P. K. Anokhin, recalls Pavlov's style of work.

> "It was striking that he could not work for a single minute without a finished working hypothesis. Like a mountain climber who, having lost one point of support, instantly seeks for another, Pavlov would immediately start building a new working hypothesis on the ruins of the previous one that would fit the latest facts more closely... But to him a working hypothesis was only a step up to a higher level of research, and therefore he never allowed it to become a dogma. Sometimes, immersed in thought, he would change ... his hypotheses with such speed that it was difficult to keep up with him."

Hypotheses advanced in judicial investigations are called *versions*. Versions may be general, i. e., explain the entire crime as a whole; particular, i. e., explain some circumstances or aspects of the crime, or individual, i. e. explain individual facts, such as who carried out the crime, who organised it (if there were several participants), etc. For example, there are to this day various versions being advanced for the murder of US President John F. Kennedy. The general version is that which explains the crime in its entirety; there may be several particular versions: was the President killed by a lone maniac, or was it a conspiracy, what were the reasons for the murder, how were the preparations made; individual versions include: what was the weapon used to kill the President, who shot, from what premises was the gun fired, etc.

§ 2. The Construction of a Hypothesis and Stages in Its Development

Hypotheses are constructed when it becomes necessary to explain a number of new facts which are not covered by previously known scientific theories or other explanations. To begin with, each individual fact is analysed, and then their totality is analysed. As already pointed out in the chapter on the logical foundations of the theory of argumentation, it is essential to examine facts only in their totality. Additional scientific experiments or experiments in the course of investigative practice are carried out in order to back up the hypothesis advanced.

The following step is a synthesis of facts and the formulation of a hypothesis. A hypothesis should not contradict scientific laws and theories which have been discovered previously and corroborated by practice. *Competing hypotheses* may be advanced to explain one and the same phenomenon in different ways, as demonstrated, for example, in the case of the origin of viruses. Hypotheses about the organic and inorganic origin of oil, etc. are competing. In constructing a hypothesis, it is essential to take into account the requirement for the hypothesis to explain as many as possible of the facts that were subjected to analysis and also that it should be as simple as possible in the form of their substantiation.

In the process of construction and corroboration, a hypothesis undergoes several stages (various authors put their number at two, three, four or five). We may illustrate these stages by constructing one of the hypotheses about the Tunguska meteorite.

First stage. *The identification of a group of facts which are not covered by previous theories and hypotheses and are to be explained by a new hypothesis.*

When the Tunguska meteorite fell, these facts were the following: The taiga in the Podkamennaya Tunguska river valley was immersed in sunlight. Suddenly a huge firy sphere fell into the valley from the sky. People who were travelling that day, June 30, 1908, on the Trans-siberian Railway spoke of a column of flame which shot up from the surface of the Earth like a fountain. The edges of the column of flame had a blue shine and reached the lower layers of the stratosphere. The explosion was accompanied by an earthquake which covered the whole of Central Siberia. Seismic waves were recorded by many geophysical stations around the world. Calculations which have already been made show that the explosion on the Tunguska was as powerful as that of a 20-megaton hydrogen bomb. The air wave it caused went twice around the globe... It is remarkable that no craters or traces of meteorite matter were found in the disaster area.

Second stage. *Formulation of a hypothesis (or hypotheses), i.e., assumptions which explain the facts in question.*

There exists not one, but at least half a dozen hypotheses about how the Tunguska meteorite came to fall to the Earth. Let us give some of them. One hypothesis assumes that a whole swarm of meteorites entered the atmosphere and fell to the Earth in the form of fiery rain. Another hypothesis suggests that it was the nucleus of a comet consisting of ice and congealed gases. Passing through the thick layers of the atmosphere, it got heated, and the gas which formed when the cosmic ice made impact with the ground shot up as a fountain of flame, giving rise to a huge fire in the taiga.

In 1973, another hypothesis was published as an interpretation of the 1908 disaster by two American astrophysicists: The Earth was pierced in a straight line by a so-called black hole (a clot of matter contracted by gravitation to a negligible volume with an infinitely high density). Although a "black hole" has a huge weight, making up an appreciable part of the Earth's mass (approximately a million billion tonnes), its diameter is hardly larger than that of a single atom. The authors suggest that this is the reason why the Earth was able to survive the collision. This hypothesis is, however, rather unlikely.

Third stage. *The derivation of all consequences following from the hypothesis in question.*

The following consequences follow from the hypothesis about the "black hole": There would be no giant crater; powerful layers of plasma would be formed around its fine trajectory as this cosmic object (a fantastic clot of matter) made its way through the Earth's air envelope; the plasma would be followed by a powerful pressure-shock front; the blue edges of the column of flame would emerge as a result of invisible X-rays being converted into visible light.

Fourth stage. *Comparison of consequences drawn from the hypothesis with existing observations, results of experiments and laws of science.*

Observations in the area where the meteorite fell showed that there really was no crater; the roar of the powerful blast extended right to Mongolia; people observed the blue edges of the column of flame.

Fifth stage. *Transformation of the hypothesis into a true piece of knowledge or a scientific theory if all consequences deriving from the hypothesis are corroborated and no contradictions arise with previously known laws of science.*

None of the hypotheses listed or others have yet been corroborated.

§ 3. Means of Corroborating Hypotheses

1. The most effective way of corroborating a hypothesis is by locating the object, phenomenon or property which is the cause of the phenomenon under examination.

Examples are the discovery of Neptune, the location of a number of islands in the Arctic Ocean, the discovery of artificial radioactivity, the discovery of diamonds in Siberia by the geologist M. Popugaeva, etc.

2. The main way of corroborating a hypothesis is to derive the corollaries and verify them.

Those who attributed the Tunguska meteorite to a collision between the Earth and a "black hole" suggested the following method for checking their hypothesis: if the body entered the globe at a speed of 30 kilometres per second at an angle of 30 degrees to the horizon in the area of the Podkamennaya Tunguska river and penetrated it along a straight line, it should re-emerge on the Earth's surface somewhere between Newfoundland and the Azores, where phenomena should occur similar to the disaster in Siberia. For this reason, the American physicists studied the meteorological records for this area of the Atlantic Ocean (on June 30, 1908).

A major role in verification accords to various experiments: in space, field trials in agriculture, the search for new materials, medicines, reagents, means of curing diseases in humans, animals and plants, educational experiments and other types. An experiment more often than not takes into account the influence not of one, but of many factors, so it should be planned in such a way that the result is obtained in a shorter time, more efficiently and as cheaply as possible. This is, for example, the way medicines are tried for their therapeutic effect to choose optimal treatment tactics.

In 1982, the Lenin Prize was awarded to a group of authors headed by current Soviet health minister Yevgeny Chazov. In ten years of work, they had created enzyme curative preparations with a stability hundreds, thousands and even millions of times greater than the original enzymes. The samples obtained were tested in the most severe conditions for protein, such as during heating. The experiments revealed that an enzyme which exists in its natural state for only a few seconds at a temperature of 50 degrees, is able in its new form to maintain biological activity for hours at a temperature of 80 degrees. This is the way in which stable enzyme preparations were obtained.

Then a next step had to be taken from experiments in physical chemistry to biological ones. The choice made was streptokinase, a ferment that is capable of destroying a thrombosis. A preparation (streptodekase) was produced which is able to cure a myocardial infarction and is used in the pre-infarction stage. Streptodekase quickly dissolves clots of blood in the eyes and helps to restore sight (some 100 patients have already been cured by this method).

This new medicine, which has no equivalent anywhere in the world, has been in use since 1981.

In judicial practice, too, a major role accords to experimentation. In this context, experiments are carried out with a view to corroborating versions advanced to explain a crime.

Inference from the assertion of the consequent to the affirmation of the antecedent is a probable one, and the formula $((a \to b) \wedge b) \to a$ is not a law of logic. But this is the way to corroborate a hypothesis through the verification of its corollaries. This means that the entire totality of the interrelated corollaries must be considered, and then this hypothesis will unambiguously result only from the totality of the corollaries in question. The conclusion will not be possible, but necessary, and accord with the formula $H \leftrightarrows (C_1 \wedge \wedge C_2 \wedge C_3 \wedge \ldots \wedge C_n)$, where H is the hypothesis and $C_1, C_2, C_3 \ldots C_n$ are the corollaries deriving from it; "$\leftrightarrows$" signifies implication from the hypothesis to the totality of corollaries and vice versa. For example, H may be the supposed illness (doctor's diagnosis) and C_1, C_2, C_3 the symptoms attributable exclusively to this illness. In this case, the hypothesis will be corroborated, i. e., the diagnosis will be correct.

The two methods shown are direct methods of turning a hypothesis into a true piece of knowledge.

3. The indirect method of turning a hypothesis into a true piece of knowledge involves disproving all false hypotheses (or versions), which then allows the conclusion to be drawn that the one remaining assumption is true. For this purpose we use the disjunctive-categorical inference and the negative-affirmative modus. Its structure is the same as in the case of indirect proof.

Pattern

Phenomenon *A* could have been caused either by *B*, or *C*, or *D*.
Phenomenon *A* was not caused by *B* or *C*.

Phenomenon *A* was caused by *D*.

Two conditions must be satisfied in this connection: first, all possible hypotheses must be listed, with disjunction either exclusive or inclusive; second, all false hypotheses should be disproven. The indirect method of corroborating a hypothesis may be used in investigative practice to give a necessary conclusion.

§ 4. Disproving Hypotheses

Hypotheses are disproved by disproving (proving the falsity of) their corollaries. It may be observed that many or all of the corollaries of the hypothesis in question do not occur in reality. It is also possible that facts will be found to contradict the derived corollaries.

A hypothesis is disproved in the negative modus (*modus tollens*) employing a conditional-categorical inference with the form: $((a \rightarrow b) \wedge \bar{b}) \rightarrow a$. This modus always provides a necessary conclusion.

The structure of the disproof of a hypothesis is as follows:

> If there was cause (hypothesis) H, then there must be the corollaries: C_1, C_2 and C_3 ... and C_n.
> The corollaries C_1, or C_2, or C_3 ... or C_n are absent.
> ______________________________
> There was no cause H.

In symbolic logic, this inference may be written as follows:

$$H \rightarrow (C_1 \wedge C_2 \wedge C_3 \wedge \ldots \wedge C_n)$$

$$\overline{C_1 \wedge C_2 \wedge C_3 \wedge \ldots \wedge C_n}$$

$$\overline{}$$

$$H$$

In this inference, we use De Morgan's law $\overline{a \wedge b \wedge c} \equiv \equiv \bar{a} \vee \bar{b} \vee \bar{c}$, in which the disjunction is taken to be inclusive. This means that one, two, three or all n corollaries may be absent. For this reason, the following notation of the disproof of a hypothesis by disproving (proving the falsity of) its corollaries will make it clearer and easier to apply:

$$H \rightarrow (C_1 \wedge C_2 \wedge C_3 \ldots \wedge C_n)$$

$$\bar{C}_1 \vee \bar{C}_2 \vee \bar{C}_3 \vee \ldots \vee \bar{C}_n$$

$$\overline{\phantom{\bar{C}_1 \vee \bar{C}_2 \vee \bar{C}_3 \vee \ldots \vee \bar{C}_n}}$$

$$\bar{H}$$

When expressed more precisely, this structure coincides in terms of its formula not with the *modus tollens* rule, which has only one antecedent and only one consequent, but with a simple destructive dilemma, or trilemma, or polylemma, depending on how many corollaries derive from the hypothesis in question, i. e., two, three or more.

Let us give an example of the disproof of a hypothesis from which six corollaries derive, i. e., an example of a simple destructive polylemma.

> If a patient is suffering from membranous pneumonia, he will have a high temperature (39-40°), chill, frequent dry coughing, pains in the side, shortage of breath and he is generally in a bad state.
>
> The patient in question does not have a high temperature (39-40°), or a high fever, or he has no frequent dry coughing, or he has no pains in the side, or he is not short of breath, or the patient's state is not generally bad.
>
> ---
>
> This patient is not suffering from membranous pneumonia.

The greater the number of corollaries lacking, the higher the degree of disproof of the said hypothesis. If in the example given only one or two corollaries had been missing, it would not have been possible to conclude that the patient was not suffering from membranous pneumonia. In this connection, the corollaries to be disproved (i. e., their falsity to be proved) should be taken as far as possible in their totality. However, the simple absence of corollaries (or an impossibility to find them) does not entirely disprove the hypothesis. Since at the time in question or in the given circumstances we were not able to trace them, the hypothesis (or version) advanced is subject to doubt. A hypothesis is definitively disproved if facts, circumstances or phenomena crop up which contradict the corollaries derived from the said hypothesis.

§ 5. The Role of Hypotheses in Cognition

The laws of science and theory all went through the stage of being hypotheses and, for this reason, a lecturer presenting theories from the domain of natural science should also depict the stages preceding proof of the theory. Scientists have repeatedly referred to the enormous role played by hypotheses. Lomonosov wrote that hypotheses were the only way by which great people came to discover the most important truths. Examining the role of hypothesis in the development of natural science, Engels wrote: "Further observational material weeds out these hypotheses, doing away with some and connecting others, until finally the law is established in a pure form. If one should wait until the material for a law was *in a pure form*, it would mean suspending the process of thought in investigation until then and, if only for this reason, the law would never come into being."[1]

Students should be shown the huge effort great scientists devoted to collecting scientific facts and also to systematising them when constructing and corroborating scientific hypotheses. The history of scientific thought facilitates a higher academic level in the teaching of every subject.

Having in mind many sciences, Engels wrote that "things are even worse with astronomy and mechanics, and in physics and chemistry we are swamped by hypotheses as if attacked by a swarm of bees".[2] Physics and chemistry teachers can use a large amount of interesting material from their disciplines to illustrate this thought of Engels on the role of hypotheses in cognition. We shall cite just a few illustrations.

Konstantin Tsiolkovsky was the founder of the theory of space flight. In 1903, Tsiolkovsky published his outstanding work *Research into the World's Expanses Using Rocket Devices*, which, according to Academician Sergei Korolyov, was crucial in determining the path he took in life and science. In this work, Tsiolkovsky formulated the hypothesis: "Centrifugal force balances out gravity and reduces it to zero" and described this as the key to space flight. "Calculations could also indicate the speeds necessary to escape the Earth's gravity and reach the planets."[3] (Let us remind readers that Tsiolkovsky used the result of mathematical calculations as

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[1] F. Engels, *Dialectics of Nature*, Progress Publishers, Moscow, 1972, p. 240.

[2] F. Engels, *Anti-Dühring*, Progress Publishers, Moscow, 1975, p. 104.

[3] K. E. Tsiolkovsky, *Works on Cosmonautics*, Moscow, 1967, p. 126 (in Russian).

his facts.) "Almost all the sun's energy is presently lost without any benefit to man, for the Earth receives two (more precisely 2.23) billion times less energy than the sun emits.

"What is strange about the idea of using this energy? What is strange about the thought of conquering the limitless space surrounding the globe..."[1]

The scientific information that has resulted from the peaceful uses of space, and also the idea of heliopower stations, which scientists believe will be able to compete with thermal and atomic ones, were also just hypotheses at one time.

The theory of natural radioactivity is also of interest. In 1903, Antoine Henri Becquerel, Pierre Curie and Marie Curie-Sklodowska were awarded the Nobel Prize for the discovery of radioactivity (the natural radioactive elements polonium and radium). Following four years of hard work, which included the manual processing of over a tonne of uranium ore, Marie Curie managed to separate pure radium chloride. In 1911, Marie Curie was awarded the Nobel Prize for chemistry for having obtained metallic radium (together with André Louis Debierne). She is the only woman ever to receive two Nobel Prizes. Marie Curie wrote that the study of the physical properties of radioactive substances had not yet been completed and, although some of the most important propositions had been established, there was still a large measure of guesswork involved. The investigations of the various scientists studying these substances differed as often as they coincided. These statements by Marie Curie bear witness to hypotheses ("guesswork") and the appearance of competing hypotheses when scientists' opinions frequently did not accord with one another.

There are currently a whole series of hypotheses being advanced about the possibility of devising a universal theory to describe all physical phenomena, including the microworld, macroworld and megaworld. But that is a matter for the future.

Numerous hypotheses have been advanced and cor-

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[1] *Ibid.*, p. 129.

roborated in the history of chemistry. A classical example is the outstanding way in which the periodic law and the periodic system of chemical elements by Dmitry Mendeleyev was corroborated. The result was the prediction of elements that were still unknown at that time, and that uranium, thorium, beryllium, indium and a number of other chemical elements should have their own atomic weights. This forecast was subsequently corroborated empirically. Mendeleyev also devised other hypotheses: about chemical energy, the limit of chemical compounds, the structure of silica compounds, etc.

Many hypotheses are put forward in the sciences which study organisms. In his research into the origin of species Charles Darwin relied on hypotheses based on the generalisation of a large number of facts he collected during his five-year expedition on *The Beagle.*

Carolus Linnaeus walked almost 7,000 kilometres during his trips over northern Scandinavia, studying the area and collecting factual material to construct hypotheses and his artificial classification of plants. He visited many European countries and surveyed the herbariums of many botanists. His disciples went to Canada, Egypt, China, Spain and Lapland, from where they sent him the plants they had collected. Linnaeus' friends from various countries sent him seeds and dried plants. This was the immense material on which Linnaeus based his systematisation.

In studying the work of Ivan Pavlov, we see how his factual material grew and was corrected, how his ideas gradually emerged about the various aspects of an object and, finally, how he increasingly formed a general picture of higher nervous activity.

Also of interest is the work carried out by Louis Pasteur on problems of wine spoilage, as a result of which he devised the biochemical theory of fermentation. One of the corollaries of this theory was the development of the process which later became known as pasteurisation. Pasteur's study of the silkworm disease was likewise of enormous practical value. As a result of this disease, over 3,500 silkworm breeders in the silk-producing areas of France found themselves ruined. Pasteur devoted almost five years to arduous experiments, sacrificing his health in the process, but nevertheless considered himself fortunate to have brought such benefit to his country by seeking out ways of preventing such devastation: "It is a matter of honour for a scientist when disaster strikes to sacrifice everything in a bid to help overcome it. It may be for this reason that I set a beneficient example to young scientists in making prolonged efforts to resolve a difficult and ungratifying task."[1]

Alongside these classical examples of how hypotheses became theories as a result of their corroboration, biology teachers can also illustrate contemporary biological hypotheses. Students' attention should be drawn to the fact that many of them are constructed on an interdisciplinary basis. The hypothesis referring to the possibility of obtaining considerable harvests on saline soils is of great importance, since they take up some 10 million square kilometres throughout the world, and the total world arable land currently amounts to 15.5 million square kilometres, i.e., saline soils account for a major portion of all land. The problem of how to turn barren saline soils into fertile agricultural land is therefore a worldwide issue. One of many hypotheses which have been advanced is the suggestion that halophytes, plants resistant to salt, be grown on these soils. As the means offered by genetic engineering multiply, such proposals will increase and we may expect considerable success in the systematic changing of many animal and plant species.

We have presented hypotheses from various domains of natural science. A large number of varied hypotheses are also put forward in the social sciences. Lenin commented on Marx's development of the materialist concept of history: "Then, however, Marx, who had expressed this hypothesis in the forties, set out to study the factual ... material. He took one of the social-economic formations – the system of commodity production – and on the basis of a vast mass of data (which he studied for not less than twenty-five years) gave a

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[1] Oeuvres de Pasteur, Vol. IV. Études sur la maladie des vers a soie. Paris, Masson et C[ie] Editeurs, 1926, p. 7.

most detailed analysis of the laws governing the functioning of this formation and its development." "Now—since the appearance of *Capital*—the materialist conception of history is no longer a hypothesis, but a scientifically proven proposition."[1] Lenin repeatedly stated that the materialist conception of history was of fundamental importance to the development of the social sciences and that this hypothesis had for the first time raised sociology to the status of a science. Lenin had the following to say on the vital significance of the materialist conception of history:"...only the reduction of social relations to production relations and of the latter to the level of the productive forces, provided a firm basis for the conception that the development of formations of society is a process of natural history. And it goes without saying that without such a view there can be no social science."[2] This brings us back to Engels' proposition that science develops through the advancement of hypotheses. However, hypotheses also have a practical significance.

We mentioned above the practical role of hypotheses in jurisprudence. Any investigation of a crime calls for the advancement of all possible versions to explain the crime and their checking.

Hypotheses about more effective ways of conducting the teaching process are also advanced and experiments carried out in schools to corroborate these hypotheses in relation to educational science, most notably the methodology applied to the teaching of mathematics, physics, chemistry and other subjects.

The examples given (both classical and contemporary) to illustrate hypotheses in physics, chemistry, biology and history show quite clearly that a hypothesis is a form of developing knowledge in all disciplines, and also in all other (and not just scientific) domains of human thought.

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[1] V.I. Lenin, "What the 'Friends of the People' Are and How They Fight the Social-Democrats", *Collected Works*, Vol. 1, 1963, pp. 141, 142.

[2] *Ibid.*, pp. 140–141.

Chapter VIII

STAGES IN THE DEVELOPMENT OF LOGIC AS A SCIENCE AND THE MAIN TRENDS IN MODERN SYMBOLIC LOGIC

§ 1. A Brief Historical Survey of Classical and Non-Classical Logic

Logic initially emerged and developed within the heart of philosophy, which was a unified, integral science uniting all the knowledge available on the objective world and about Man himself and his thinking. At this stage in its historical development, logic was primarily ontological in character, i.e., it identified the laws of thinking with those of being.

In the beginning, the laws and forms of correct thinking were studied within the framework of the art of oratory, this being one means for influencing people's minds and convincing them of the expediency of a particular form of behaviour. This was the case in Ancient Greece, Ancient India, Ancient China, Ancient Rome and Mediaeval Russia. In the art of eloquence, however, the logic aspect was still a subordinate one, since logical means served to convince the audience, rather than attain the truth.

The development of the science over several centuries followed two channels that were entirely separate. One of these trends in logic sprang from Ancient Greek logic (Aristotle in particular), which provided the basis for the development of the logic in Ancient Rome, then in Byzantium, Georgia, Armenia, the Arab countries of the Middle East, Western Europe and Russia. The other trend derived from Indian logic, from which sprang the logic of China, Tibet, Mongolia, Korea, Japan, Indonesia and Ceylon.[1]

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[1] *See*: A.O. Makovelsky, *A History of Logic*, M., 1967. In this paragraph, the author also used information taken from the book by the well-known Soviet historian of logic N.I. Styazhkin *The Formation of Mathematical Logic*, Moscow, 1967 (in Russian).

Logic in Ancient India

The history of Indian logic is connected with the development of Indian philosophy. The most ancient works of literature still extant in India are the Vedas (2nd and early 1st millennia B.C.), the oldest part of these being the Rig-Veda. The Upanishads, explanations of the Vedas, were treatises written in prose by the Brahmins to develop and comment on many of the philosophical ideas contained in the Vedas.

The Indian scholar Madhava, in his essay *Compendium of Speculations* (1350 A.D.) lists 16 schools of Ancient Indian philosophy. The most important was the materialist philosophy school of Charvaka (the founders were Brihaspati and his student Charvaka). A related school was the Lokayana. Basically materialistic were the rationalist philosophical systems: Vaisesika (its founder was nicknamed "Kanada" which means "atom eater"), Nyaya (founded by Gautama) and Jinaism (founded by Vardhamana Mahavira, who came to be called Jina, the victor). Materialism as a philosophical trend proceeds from the assumption that the world is material, exists objectively, outside and irrespective of the consciousness, that matter is primary and eternal, while consciousness and thinking are a property of matter.

There were also idealistic philosophical systems in Ancient India that maintained the primary nature of the mind, consciousness and thinking. The main ones were: Yoga, Mimansa, Vedanta and Buddhism. The leading philosophical systems also included Sankhya, a dualist system proceeding from recognition of the equality of the two principles – the spirit and matter, the ideal and the material.

Disputes between representatives of the various philosophical schools promoted the development of the theory of cognition and logic, but logic is interpreted independently only by the Nyaya school, though not yet systematically, but in the form of short aphorisms (Sutras). Only beginning with the Dignaga (6 c. A.D.) did Indian logic acquire an elegant and systematic form.

Indian logic developed over two millennia and the history of this development has not yet been thoroughly studied. Although the bibliography on Indian philosophy and logic is enormous, there is, as yet, no unanimous opinion concerning the course of its development.

Indian logic focuses considerable attention on the theory of inference, which it identifies with proof. The original view that syllogism consists of ten propositions (members) changes. As logic developed, the members of syllogism tended to be reduced. Gautama cut them down to five: 1) thesis, 2) basis, 3) example, 4) application and 5) conclusion. This system of syllogism predominated in Indian logic.

Indian logic has the following specifics:

1) an original teaching on the five-membered syllogism, in which the idea of the inseparable link between deduction and induction is important;

2) proposition is not recognised as an independent act of thought, but is regarded as a member of inference;

3) perception is not something directly given, but includes an act of "proposition-inference". In other words, our perceptions are based on the experience we have acquired;

4) a distinction between speech "for oneself" (i. e., internal speech, which constitutes a form of the thinking process when a person converses, as it were, with himself) and speech "for others" (i. e., external speech, when a person transmits ideas and communicates with other people in an oral or written form). The former is characterised by a briefer mode of thinking than the latter. It should be noted that only in the 20th century did European psychology begin studying these types of speech and establishing the differences between them.

The systems of Indian logic (the "old" Nyaya, Buddhist logic and the "new" Nyaya) are set out in condensed form in the two-volume *Indian Philosophy* by Sarvepalli Radhakrishnan.

One of the fullest systematic presentations of the foundations of the Indian Navya-Nyaya logic is given by Daniel Ingalls, a Harvard lecturer and an eminent American Indologist.[1]

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[1] Daniel Henry H. Ingalls, *Materials for the Study of Navya-Nyaya Logic*, Cambridge (Mass.)–London, 1951.

Navya-Nyaya (the "new method", "new logic") is the only complete system of logic to have emerged outside European culture. The school is considered to have been founded by Gangesa (12-13th c.), who wrote the *Tattvacintamani* treatise. In this school logic becomes an independent science, as well as a method and instrument for scientific cognition. Yet this system, too, has its defects: a cumbersome system of categories in the ancient tradition, non-observance of the difference between an abstract conclusion and a concrete example of a conclusion. These are largely overcome by the later, or radical school of Navya-Nyaya, founded by Raghunatha.

In his account of the chief concepts, theories and methods of the Navya-Nyaya logic, little known outside India, and of the leading representatives of this school from the 12th to 17th centuries, Ingalls relies on the achievements of contemporary symbolic logic.

From its initiation until the 1920s, logic developed primarily in the direction of formalisation and cataloguing of the correct methods for reasoning within the framework of the two values of truth. Propositions could be either true or false. This *logic* was called *classical*, since it relied on ancient tradition. It is also called traditional or two-valued logic. Classical logic is the first stage in the development of formal logic.

As scientific knowledge expanded, logic rose to a second, higher stage of development, when it systematised forms of thinking, applying mathematical methods and a special range of symbols. By studying conscious thought with the help of calculus, it goes further towards abstraction. This formal logic is called symbolic or mathematical logic, but it is still classical in that it continues to operate with two values of truth.

Within modern mathematical logic, *non-classical logics* are developing. These operate either with an infinite set of truth values or with constructive (compared with classical logic) methods for proving the truth of propositions, or modal judgements; or they exclude the negations found in classical logic.

Ingalls notes in his book that formal Navya-Nyaya logic is distinguished by a high degree of abstraction. Its followers did not confine themselves to purely linguistic analysis; they were always striving to disclose the relations between things themselves. In some respects, the American scholar believes, the Navya-Nyaya is superior to Aristotle's logic. Its creators had, for example, an idea of conjunction, disjunction and their negation and knew the corollary on classes from De Morgan's law. In the Navya-Nyaya school, the quantifiers, i. e., logical terms expressed by the words "all', "some", "any" and the like were hardly ever used, since they were expressed with the help of abstraction of properties and by means of combining negations. The Navya-Nyaya analysed the following problems: the relationship of "penetration" (i. e., the theory of logical sequence), the problem of negative propositions, modes of forming complex terms, and others.

The Navya-Nyaya never got round to using symbols, though in Ingalls' opinion this can hardly be considered a shortcoming. After all, no one, with the exception of the Stoics, used symbols in logic up until the 19th century. Instead of symbols, a complex system of clichés was elaborated here, and thanks to these multitude of expressions was attained. In the formal logical system under consideration, Ingalls is inclined to see the rudiments of a number of ideas that have been developed in mathematical logic.

Ancient Indian logic was original. It emerged and developed independently of that of Ancient Greece. India became aware of Greek philosophy and logic only as a result of Alexander the Great's (356-323 B.C.) invasion.

Logic in Ancient Greece before Aristotle

In Ancient Greece, we encounter a logical form of proof in the form of a chain of deductive inferences in the Eleic school (Parmenides and Zeno). Heraclitus of Ephesus offered a teaching on universal movement and change. The struggle between these two philosophies in Ancient Greece was really between the metaphysical trend in philosophy (considering phenomena as constant and independent of one another) and the dialectical (when phenomena of reality were cognised in their development and self-movement).

In the mid-5th c. B. C. the so-called Sophists appeared in Ancient Greek philosophy (Protagoras, Gorgias and others). The main subject of their philosophical studies was not Nature (as it was before them), but Man and his activities, including ethics, rhetoric, and grammar. Protagoras, Gorgias and Thrasymachus were the first in Greece to create a theory of rhetoric. The Sophists criticised both religion and materialist philosophy. In elaborating a theory of eloquence, they touched on questions of logic. Protagoras wrote a special essay entitled *The Art of Debate*. He was himself a master of debate, travelling around Greece and organising debates that drew large audiences. In the words of the Ancient Greek author Diogenes Laertius, "he was the father of the whole tribe of eristical disputants now so much in evidence".[1]

Protagoras was the first to use the "Socratic mode of discussion", which consisted in setting the other person questions and demonstrating the error of his answers. For this reason, Protagoras began to study forms of inference in the speech of orators on the plane of logical methods. This was also done later by Aristotle in his *Topics*. Protagoras' essay "Disquisitio de Protagorae irta et Philosophia" was dedicated to a well-known sophism concerning the debate between Protagoras and his pupil Euathlus.

The Sophists were opposed by the outstanding materialist of Ancient Greece Democritus (460-370 B.C.), who created an all-embracing philosophical system that included a teaching on being, cosmology, a theory of cognition, logic, ethics, politics, aesthetics and a number of other spheres of scientific knowledge: mathematics, physics, biology, medicine, philology and others. Democritus was the creator of the first system of logic in Ancient Greece. He wrote the three-volume treatise *On Logic, or Criterion of Thought*. Unfortunately, only very small fragments of this work are still extant. In his work *On Logic*, Democritus builds logic on an empirical basis, so he was one of the founders of inductive logic.

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[1] Diogenes Laertius, *Lives of Eminent Philosophers*, Vol. 2, London, William Heinemann, Ltd; New York, G.P. Putnam's Sons, 1931, p. 465.

He considered propositions dividing them into subject and predicate, and also the definitions of concepts.

The *Criterion of Thought* set out Democritus' teaching on the kinds of knowledge, in which he did not separate questions of logic from the theory of cognition. Successors of Democritus were the philosophers of the Epicurean school. This trend in logic anticipated the inductive logic of Francis Bacon and was counterposed to the idealistic Socratean-Platonic logic.

The Ancient Greek philosophers Socrates (about 469-399 B.C.) and Plato (428-347 B.C.) were also engaged in problems of logic. Socrates emphasised the problem of the methods by which true knowledge could be obtained. He believed that any object could be cognised only if it were reduced to a universal concept and judgements made about it on the basis of this concept. For this reason, he suggested that his opponents in a debate define such concepts as "justice", "injustice", "bravery", "beauty" and the like. They always gave superficial, unconsidered definitions. Taking instances from daily life, Socrates would show that the given definition was mistaken or insufficient and would lead his interlocutor to correct it. The new definition was retested, supplemented, and so on. For instance, when defining "injustice", such actions as lying, deceit, doing evil, enslaving, and the like, were called unjust. Later it was revealed, however, that, during a war these actions do not fall within the concept of injustice. The initial definition is then restricted: these actions are only unjust in relation to friends and allies. Yet this new definition is again inadequate. After all, someone who deceives his sick child into taking a medicine or takes the sword away from a friend trying to commit suicide is not committing an injustice. Consequently, only he who does something towards friends with the intention of doing them harm is committing an injustice towards them.

Socrates understood knowledge as a judgement concerning something common for a whole series of things (or their properties). Knowledge is thus the concept of an object and is attained by defining this concept. Moreover, it is judged as the similarity or common nature of the objects included in the given concept and as the distinction between that which falls within the given concept and that which comes under a similar or related one. Socrates' teaching on knowledge as the definition of universal concepts and the inductive methods he used for defining ethical concepts played a marked role in the development of logic.

Socrates' teaching on knowledge was developed by his pupil Plato in the theory of "forms" or "ideas". He created the system of objective idealism, which maintained the existence of the spiritual basis external to and independent of the human consciousness. Plato founded his school in Athens, where he set up an Academy. He turned Socrates' universals into absolute ideas that exist in themselves, outside the cognising subject and independently of the material world. He also considered these ideas to be primary, eternal and constant, forming a special other world. The material world, according to Plato, was secondary; it was changeable and reflected the eternal, constant ideas that were the prototypes of all existing material things, while these things were only the "shadows" of ideas.

Plato focused considerable attention on questions of the theory of cognition and logic. He strove to form a concept and then divide it into types, the favoured method being dichotomy, i.e., the division of the concept *A* into *B* and not-*B* (for example, animals are divided into vertebrates and non-vertebrates). He formulated two rules for dividing concepts, and developed the theory of judgement in the dialogue *Sophist*. Plato distinguished between the relationsip of distinction and that of opposition.

Plato's school was much concerned with defining, particularly, the objects of organic and non-organic nature. The following definition of Man was Plato's: "Man is a two-legged animal without feathers". Hearing this, Diogenes plucked a cockerel, and let it go during one of Plato's lectures at the Academy saying "There is Plato's Man". Plato admitted his error and introduced a correction into his definition: "Man is a two-legged animal without feathers and with broad nails".

One of the greatest scholars and philosophers in the Ancient World was *Aristotle* (384-322 B.C.). He was born in the city of Stagira, so he is called the Stagirite. Aristotle's profound works are devoted to diverse branches of contemporary knowledge: philosophy, logic, physics, astronomy, biology, psychology, ethics, aesthetics, rhetoric and other sciences.

Over a period of twenty years, Aristotle was a pupil of Plato's school. Twelve years after Plato's death, Aristotle founded his own school of philosophy in Athens (the Peripatetic school). He wrote a total of about one thousand works.

Aristotle gave the first systematic presentation of logic. His logic is called "traditional" formal logic, and it includes such sections as concept, proposition, laws (principles) of correct thinking, inference (deductive, inductive, by analogy), the logical foundations of the theory of argumentation, and hypothesis. Aristotle's chief works on logic were *Prior Analytics* and *Posterior Analytics*, in which he presents the theory of the syllogism, the definition and division of concepts, the theory of proof. Aristotle's other works on logic include *Topics*, containing his teaching on probable "dialectical" proofs, *Categories, On Sophistical Refutations* and *On Interpretation*. The Byzantine logicians later combined all these works by Aristotle under the common title of *Organon* (tool of cognition).

Aristotle also set out the laws of correct thinking – the law of identity, the law of non-contradiction, and the law of the excluded middle – in his major work *Metaphysics*. Aristotle initially considered his laws of thinking as laws of being, but believed the logical forms of true thinking to be a reflection of real relations.

For Aristotle, truth is a correspondence between thought and reality. He regarded a proposition as true if the concepts it contained were combined in the same way as things are interconnected in nature. A false proposition was one that combined things that were disunited in nature, or vice versa. Relying on this concept of truth, Aristotle created his own logic. In the *Analytics*, Aristotle develops modal logic quite extensively and gives a description of syllogisms from hypotheses.

Lenin described Aristotle's logic as the movement of thought – "an inquiry, a searching ... searchings, waverings and modes of framing questions". The strength of his teaching lay in the fact that it contained "the living germs of dialectics *and inquiries* about it".[1]

Aristotle saw logic as a tool or method of research. The core of Aristotle's logic is the theory of deduction. It contains elements of mathematical (symbolic) logic and the beginnings of propositional calculus.

The further development of propositional calculus, including the theory of conditional and disjunctive inferences, was carried out by the logicians of the Megarian School of philosophy (a teaching known as the "logic of the Stoics"). The founders were Zeno (300 B.C.) and Chrysippus (281/78-208/05 B.C.). The Megarians were Diodorus Cronus, Stilpo of Megara, Philo of Megara and Eubulides of Miletus.

According to their teaching, logic should study both verbal signs and the ideas they represent. They saw the purpose of logic as learning how to judge correctly about things in order to release the mind from delusions. The Stoics divided logic into dialectics and rhetoric, thus going beyond the restricting bounds of formal logic.

Unfortunately, only small fragments of the logical teachings of the Megarians and Stoics are still extant. The logicians of this school analysed logical terms: negation, conjunction, disjunction, implication. As a result of the discussion on implication, they developed four different understandings of it. The Megarian Eubulides discovered the first semantical paradox we know in history, called "the Liar".

Mediaeval logic

Mediaeval logic (6-15th centuries) has not been adequately studied. In the Middle Ages, the struggle between materialism and idealism in logic centred mainly on the interpretation of the nature of the universals. The so-called realists, continuing Plato's idealistic line, believed that universals exist in reality,

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[1] *See:* V.I. Lenin, "Conspectus of Aristotle's Book *Metaphysics*", *Collected Works*, Vol. 38, pp. 366, 367.

outside and independently of individual things. The nominalists, on the contrary, believed, that only individual objects really exist, while universals are merely names for them. Both views were incorrect, though nominalism was closer to materialism.

Let us formulate the chief problems that were elaborated in mediaeval logic: those of modal logic, analysis of distinguishing and excluding propositions, the theory of logical sequence, the theory of semantical paradoxes (logicians in the Middle Ages were actively engaged in analysing these – for example, the paradox of "the Liar" and others, and proposed diverse solutions).

The theoretical sources of mediaeval Arab logic should be sought in Aristotle's logic. The founder of Arab logic is considered to be the Syrian mathematician Al-Farabi (c. 870-950 A. D.), who commented on the whole of Aristotle's *Organon*. Al-Farabi's logic was oriented on analysing scientific thinking and it also studied questions of the theory of cognition and grammar. Like Aristotle, he correlates the method of thinking with real existential relations. Aristotle was Al-Farabi's spiritual father in the sphere of logic. The Arab scholar distinguishes two stages in logic: the first embraces ideas and concepts, the second – the theory of propositions, conclusions and proofs.

Syrian logic was a connecting link between Ancient and Arabic science. The historians of logic recognise the influence of Arabic logic on the development of European logic in the Middle Ages.

Avicenna (Ibn Sina) (c. 980-1037 A. D.) comments on Aristotle and himself attempts to develop logic. He elaborated the dependence between categorical and conditional propositions, an expression of implication through disjunction and negation, i. e., the formula $(p \rightarrow q) \equiv (\bar{p} \vee q)$. In the textbook *Logic*, Avicenna was striving to generalise Aristotle's syllogistics. Initially, he used Al-Farabi's commentaries on Aristotle's *Metaphysics*.

Another major Arab Aristotelian was Averroës (Ibn Rushd) (1126-1198), who also made detailed comments on Aristotle's logical texts and developed the understanding of modality.

In the second half of the 13th century, the most

18-570

popular book on logic was *Summulae logicales*, by Peter of Spain (Petrus Hispanus) (c. 1220-1277). Logic was also developed by the British scholar John Duns Scotus, the Spaniard Raymond Lull, the Englishman William of Ockham, the Frenchman Jean Buridan, and the German Albert of Saxony. Peter of Spain's treatise contains a number of new ideas (compared with the Megarian-Stoic school) concerning the logic of propositions.

In the 15th and 16th centuries, in the *Age of Renaissance*, the empirical trend in logic and the methodology of scientific knowledge gained strength. Science was developing rapidly, the great geographical discoveries were being made, and science was drawing closer to practice. Mathematics began to play an ever increasing role in other sciences.

A major contribution was made to the elaboration of the materialistic foundations of logic by Francis Bacon (1561-1626), who was the founder of English materialism. Bacon opposed extremes of rationalism and empiricism, saying that the scholar should not become either like a spider, spinning a web out of himself, or an ant that merely collects and accumulates material, but should, like a bee, collect and then process material, transforming it into scientific theory.

Bacon elaborated the principles of inductive logic in his famous work *The New Organon*. As the title itself indicates, Bacon counterposes his logic to that of Aristotle. His *The New Organon* was to replace the old *Organon* of Aristotle. But Bacon was unjust towards Aristotle as he did not know the true Aristotle, but became acquainted with his works as set out by mediaeval philosophers. Bacon elaborated questions of scientific induction, the aim of which is to reveal causal relations between phenomena in the surrounding world. He also developed methods for determining these relations: the method of similarity, the method of distinction, the combined method of similarity and distinction, the method of accompanying changes, the method of residuals. Questions of scientific induction were further developed in the 19th century by John Stuart Mill and other logicians.

The French philosopher René Descartes (1596-1650) formulated four rules by which the scholar should be governed in the course of any scientific research. In 1662, his followers Antoine Arnauld and Pierre Nicole wrote the book *The Art of Thinking (The Port Royal Logic)*, which set out to relieve Aristotle's logic of the scholastic distortions introduced into it by later logicians.

The German philosopher Immanuel Kant (1724-1804) approached logic from idealistic positions. He completely isolated logical forms and laws from their content, proclaiming them to be "a priori" (i. e., preceding experience and independent of it).

Over a number of years, Kant lectured in formal logic at the university of Königsberg. One of his students worked over the notes he made during Kant's lectures and, in 1800, during Kant's lifetime, published them, but this publication cannot be considered as Kant's own work.

In Kant's definition, logic is the science of essential laws and rules of "reason in general", so, according to him, logic should study the form of thinking in isolation from its content, i.e., independently of the subject of thinking. Kant believed that logic abstracts from any content of knowledge and, consequently, from things themselves. He suggested that, after Aristotle, logic could not be enriched in content, but only improved in precision, determinacy and clarity, so he did not consider the traditional logic to be enough for the purposes of cognition and developed a transcendental logic. In his opinion, this was to overcome the limited way in which ordinary, general logic viewed the forms of thinking.

Engels unmasked Kant's view on the constancy of logic, saying that "the theory of the laws of thought is by no means an 'eternal truth'..."[1] Aristotle's logic differed fundamentally from that of Kant, for the latter's was purely subjective and totally formalistic, and its philosophical basis was subjective idealism.

Kant made a positive contribution to logic by distinguishing between logical cause and logical consequence, on the one hand, and real causes and real effects, on the other.

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[1] F. Engels, *Dialectics of Nature*, p. 43.

The major German philosopher and objective idealist Georg Hegel (1770-1831) provided an elaborate critique of Kant's formalism, including on questions of logic, but this critique was made from positions of idealistic dialectics. Hegel's logic coincides with dialectics, so he criticised formal logic and rejected it. Speaking of the reflection in movement of the concepts of movement of the objective world, Hegel understood the objective world idealistically, i.e., as the other-being of the absolute idea. Hegel gave a critique of the laws of formal logic in the second book of his work *Science of Logic*, in the section headed "Lehre vom Wesen".

Hegel's philosophy contains a rational core–his teaching on dialectics. He developed the problems of the dialectics of thought and dialectical logic.

The materialist trend in logic was continued by the Russian materialist scholars. The Russian logicians, such as P. S. Poretsky and E. L. Bunitsky, made a substantial contribution to the development of logic on the level of worldwide logical conceptions.

The first treatise on logic appeared in Russia in the 10th century. This was a translation of the philosophical chapter from the *Dialectics* of the Byzantine writer of the 7th century Ioannus of Damascus, which set out the works of Aristotle and his commentators. The first systematic textbook on logic, including Aristotle's logic and certain of Hobbes' ideas, was prepared in the second half of the 17th century. At the same time, some of the ideas of mathematical logic began to gain currency in Russia.

In the 18th century original logical contributions began to appear in Russia. The first were made by the Russian scientist of world renown Mikhail Lomonosov (1711-1765). He introduced fundamental changes into traditional syllogistics and proposed his own classification of inferences, differentiating proposition and grammatical sentence, etc. Dimitry Anichkov (1733-1788) in his treatise *Annotationes in logicam, metaphysicam et cosmologiam* studied modal judgements, dividing them into four types: necessary, impossible, possible and not impossible, and formulated a system of rules for the holding of debates.

The materialist philosopher Alexander Radishchev (1749-1802) was one of the first in world literature to raise the problem of the need for logical analysis of relations, which is not to be found in Aristotle's logic or the logic of the mediaeval schoolmen. He wrote that propositions represent a comparison of two concepts or cognition of relations existing between things. Radishchev gave the following classification of types of inference:[1] 1) "reasoning" (i. e., syllogism); 2) "equation", i. e., inference of equivalence, based on the following axiom: equal and identical things stand in an equal or identical union or relation; 3) "inference based on similarity".

The Russian revolutionary democrats, who supported emancipation of the people from exploitation [V. G. Belinsky (1810-1848), A. I. Herzen (1812-1870), N. G. Chernyshevsky (1828-1889) and N. A. Dobrolyubov (1836-1861)], were actively interested in philosophical questions, including the problems of logic. Belinsky warned against errors of logic in proving a thesis.

Herzen put forward the idea of a harmonious combination of theoretical thinking and practical activity. Chernyshevsky maintained the thesis that the concept of the relativity of knowledge did not mean that it was illusory or inobjective, but merely indicated its incompleteness.

The greatest Russian logicians of the 19th century were Mikhail Karinsky (1840-1917) and his pupil Leonid Rutkovsky (1859-1920), whose chief logical works are devoted to classifying inferences.

The main idea behind Karinsky's logical theory may be described as a striving to build an axiomatic-deductive system of logic, proceeding from the basic relationship of equivalence (i. e., "identity"), and in this to describe deductive and inductive inferences without using elements of strict formalisation. In this concept, Karinsky is close to the ideas of Jevons, as even his contemporaries noted.

The structure of inference is, according to Karinsky,

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[1] *See:* N. I. Styazhkin, V. D. Silakov, *A Short Essay on the History of General and Mathematical Logic in Russia*, Moscow, 1962, p. 15 (in Russian).

the following. From two premises with the structure (1) and (2), the conclusion (3) is drawn.

A is in relation *R* to *B*	(1)
B is identical to *C*	(2)
A is in relation *R* to *C*	(3)

Here are some examples.

Moscow is east of Paris.
Paris is the capital of France.

Moscow is east of the capital of France.

Kuibyshev is west of Lake Baikal.
Lake Baikal is the deepest lake in the world.

Kuibyshev is west of the deepest lake in the world.

Karinsky divided all conclusions into two major groups: 1) conclusions based on the "collation of subjects" and 2) conclusions based on the "comparison of predicates" (in this sense the terms subject and predicate do not coincide with their traditional meanings). The basis of the conclusion is the identity (or, correspondingly, difference) of "subjects" or "predicates". According to Karinsky, all types of inference and, in addition, the hypothesis may be included in these two big groups.

The well-known Soviet historian of logic N. I. Styazhkin came to the conclusion, following his study of Karinsky's ideas on logic, that the latter was striving for his classification to include all types of inference met in the practice of scientific and general human thinking. But the task Karinsky set himself proved broader than the premises he adopted and put at the basis of his theory. It remained unfulfilled.

Leonid Rutkovsky (1859-1920) was the author of the work *Basic Types of Inference* (1888). Whereas Karinsky tried to build a theory of inference using only the identity relation and to reduce all other relations to this, Rutkovsky believed it possible to recognise other relations, such as those of similarity, of co-existence and so on, as having stature equal to that of the relation of identity. Since a multitude of different relations exist, there are also many different types of inference. He divided inferences into intensive (i. e., considered in the logic of intension) and extensive (considered in the logic of extension).

Rutkovsky divided all inferences into two main groups. The first consists of inferences of subjects (i. e., according to extension), which break down into three types: a) traduction (inferences of similarity, identity, conditional dependence); b) induction (complete and incomplete); c) deduction (hypothetical and non-hypothetical).

The second group of inferences consists of those of the predicate (by intension), which breaks down into inferences of "production" (disjunctive syllogism, inferences on the community and contemporaneity of objects and others), "subduction" (inferences made during the classification and ordering of objects and others), and "eduction" (the inference of an object in the type of its class, inferences of mathematical probability and others).

The axiom of "production" is thus: "It follows from the fact that the object has property *B* that the same object also has property *C*, since property *B* invariably coexists with property *C*".[1]

A brief analysis of the works of Karinsky and Rutkovsky shows that their original works on the classification of the types of inference furthered the progressive development of traditional logic in the 19th century.

The logician Nikolai Vasiliev (1880-1940) from Kazan had some original ideas that followed on from his study of the problems of traditional logic. They proved so significant that they influenced the development of mathematical logic. In the footsteps of another Russian logician, S. O. Shatunovsky, he expressed the idea of the non-universality of the law of the excluded middle. Whereas Shatunovsky came to this idea as a result of a thorough study of the specifics of mathematical proof as applied to infinite sets, Vasiliev came to this conclusion through his study of the particular propositions considered in traditional logic. Vasiliev's chief works were the

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[1] L. V. Rutkovsky, "Basic Types of Inference", quoted from the collection: *Selected Works of Russian Logicians of the 19th Century*, Moscow, 1956, p. 312 (in Russian).

following: *On Particular Propositions, on the Triangle of Opposites and the Law of the Excluded Fourth* (1910), *Imaginary (non-Aristotlean) Logic* (1912) and *Logic and Metalogic*. Vasiliev supported his conceptions by formal analogy with Lobachevsky's non-Euclidean geometry. Not all Vasiliev's contemporaries appreciated his ideas, though some of them thought he had written "an extremely penetrating work". Vasiliev's logical ideas may be regarded as certain preceding thoughts, further developed in the constructive and intuitive logics on the inapplicability of the principle of the excluded middle to infinite sets. In addition, Vasiliev considers the conditions on which he believed it possible to operate with opposing propositions within a non-contradictory logical system.

In the 19th century mathematical logic appeared. Gottfried Leibnitz (1646-1716) – the outstanding philosopher and mathematician of the 17th century – is justifiably considered its founder. Leibnitz attempted to create a universal language by means of which disagreements between people could be solved through calculations. In constructing such a calculus, Leibnitz proceeded from his basic principle of reason, which was that in all true propositions, universal or particular, the predicate is always, either necessarily or by chance, contained in the subject. He wanted to give a numerical description to any concept and to establish rules for operating with these numbers that would make it possible not only to prove all logically probable truths, but also to discover new ones. He saw the latter as a special merit of his universal characteristic. Leibnitz speaks of it as a miraculous universal language with its own vocabulary (i. e., characterising numbers related to concepts) and grammar (rules for operating with these numbers). Leibnitz wanted to construct an arithmetic logical calculus in the form of some calculating machines (algorithm), but he did not succeed in this.

The most unacceptable aspect of Leibnitz's conception was the idea that the entire content of our concepts can be expressed by characteristic numbers. His idea that human thought could be completely replaced by a calculating machine was unsound.

Leibnitz suggested that mathematics could be reduced to logic, which he considered to be an *a priori* science. The advocates of this substantiation of mathematics are called logicists – representatives of the subjective idealistic trend (considering human consciousness to be primary).

Leibnitz was the forerunner of logicism in the sense that he proposed a reduction of mathematics to logic and the mathematisation of logic: the building of logic itself as some arithmetic or letter algebra. Leibnitz was also the forerunner of logicism in that he tried to create an arithmetic logical calculus, as we have said.

Let us show how Leibnitz did this. Taking the categorical syllogism

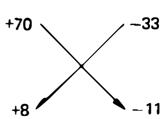
$+70, -33 \qquad +10, -3$
Every wise man is devout
$+70, -33 \qquad +8, -11$
Some wise men are rich

$+8, -11 \qquad +10, -3$
Some rich men are devout

Above the concepts a random selection of characteristic numbers for the terms of the premises is written. The truth of the universal affirmative statement "All *S* are *P*" (the first premise) is expressed so that both characteristics of the subject are divided by the corresponding characteristics of the predicate, i. e., 70 (precisely, without a remainder) is divided by 10, and -33 by -3, and the numbers along the diagonals are mutually prime, i. e., $+70$ and -3, just like -33 and $+10$ are mutually prime numbers. The truth of the particular affirmative statement must, according to Leibnitz, be expressed by the following rule: numbers on the diagonal must be mutually prime, i. e., have no common divisors apart from unity.

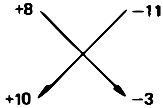
$$+70, \ -33 \qquad +8, \ -11$$

The premise "some men are rich" has the following numbers:



i.e., mutually prime numbers stand on the two diagonals.

The conclusion also satisfies this rule, for mutually prime numbers stand on the diagonals.



Leibnitz expresses the truth of the universal negative statement "No S is P" so that on at least one diagonal there are non-mutually prime numbers. The truth of the particular negative statement is expressed in that at least one of the characteristics of the subject is not divisible by the corresponding characteristic of the predicate.

In order to use Leibnitz's calculus, people had to coat their reasoning in the form of a syllogism and look to see whether it was correct or incorrect. The system Leibnitz constructed satisfied this demand only when applied to syllogisms correctly constructed according to Aristotle. The author of this textbook has shown that all Aristotle's nineteen rules of the modi of the syllogism are also correct according to Leibnitz's criteria. In relation to the incorrect modi of Aristotle's categorical syllogism, it is a different matter, however. An example can always be thought up when, given different correct selections of numerical characteristics for the premises, different assessments of the conclusion are obtained: in some cases it proves true, in others false.

Leibnitz's calculus did not, therefore, stand the test, as Leibnitz himself noted, of course, as he proceeded later to building a letter calculus according to the algebraic model. He did not succeed here either.

Leibnitz's conceptions were not totally erroneous, however. In itself, the method of arithmeticisation plays a quite considerable role as an auxiliary method in mathematical logic. It comprises, for example, the essence of the method used by the well-known Austrian mathematician and logician Kurt Gödel to prove the illusoriness of Leibnitz's dream of creating a universal characterisation making it possible to replace all human thinking with calculations.

It was precisely Leibnitz's metaphysical idea of reducing all human thinking to some mathematical calculus that was false. So its consequences were also false.

Mathematical logic was developed intensively in the works of George Boole, Ernst Schröder, William Jevons, Platon Poretsky and other logicians.

The English logician George Boole (1815-1864) developed an algebra of logic – one of the divisions of mathematical logic. He studied classes (as extensions of concepts), the correlations between them and operations on them. Boole transfers the laws and rules of algebra to logic.

In his work *An Investigation of the Laws of Thought*,[1] which exerted a considerable influence on the development of logic, Boole introduced into the logic of classes addition ("+"), multiplication ("x" or no sign) and subtraction ("–") as basic operations. In the calculus of classes, addition corresponds to the unification of classes, excluding their common part, while multiplication corresponds to their intersection. Boole regarded subtraction as the opposite action to addition – the separation of a part from the whole, which in natural language is expressed by the word "except".

Boole introduced into his system logical equations that he expressed by the sign "=", corresponding to the copula "is". He writes down "the heavenly bodies are the sun and the planets" in the form of an equation thus: $x = y + z$, hence it follows that $x - z = y$. According to Boole, in logic, as in algebra, the terms of one part of an equation can be transposed to other with the opposite sign. Boole discovered the commutative law for subtraction: $x - y = -y + x$ and the distributive law for multiplication in relation to subtraction: $z(x - y) = zx - zy$. He formulated the general rule of subtraction: "If equal things are taken from equal things, the remainders are equal. And it hence appears that we may add or subtract equations, and employ the rule of transposition above given just as in common algebra".[2]

[1] George Boole, *An Investigation of the Laws of Thought, on Which Are Founded the Mathematical Theories of Logic and Probabilities*, Dover Publications, New York, 1951.

[2] *Ibid.*, p. 36.

The subject-matter of Boole's research also included propositions (in traditional logic they are called judgements). In propositional calculus according to Boole, addition ("+") corresponds to exclusive disjunction, while multiplication ("x" or no sign) to conjunction.

In order to write down a proposition in symbolical form, Boole draws up a logical equation. If any of the terms of the proposition remains undistributed, he introduces the term $\sqrt{\ }$ to designate a class that is indeterminate in some respect. In order to express a particular negative judgement, for example, "Some people are not reasonable", Boole first presents it in the form "Some people are unreasonable" and then expresses it in symbols in the usual way.

According to Boole, there exist three types of symbolical expression of propositions: $X = \sqrt{\ } Y$ (only the predicate is not distributed); $X = Y$ (both terms, subject and predicate, are distributed); $\sqrt{\ } X = \sqrt{\ } Y$ (both terms are not distributed).

The dialectics of the correlation between assertion and negation in Boole's concepts and propositions are as follows: without negation there can be no assertion and, vice versa, any assertion contains negation. Assertions and negations are linked with the universal class: "The mind assumes the existence of a universe not *a priori* as a fact independent of experience, but either *a posteriori* as a deduction from experience, or *hypothetically* as a foundation of the possibility of assertive reasoning."[1]

Distinguishing between live, conversational language and symbolical "language", Boole stressed that the language of symbols is only an auxiliary means for studying human thought and its laws.

The German mathematician Ernst Schröder (1841-1902) collected and generalised the results obtained by Boole and his closest followers. He introduced the term "Logikkalkul" (logical calculus), and new symbols compared with those of Boole. He based the calculus of classes not on the relation of equivalence, as Boole had done, but on that of the inclusion of a class in a class, which he wrote as $A \in B$. Boole used the sign "+" to

[1] George Boole, *op. cit.*, p. 85.

designate the unification of classes, excluding their common part, i.e., symmetrical difference (see Figure 52), while Schröder used it to mean the unification of classes without excluding their common part (see Figure 10).

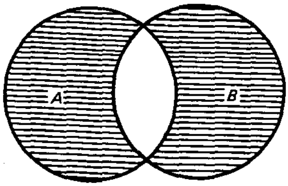


Fig. 52

By leaving out a sign Schröder implied the operation of the ordering of classes, for example *ab*. As applied to statements of the form $a + b$, it meant an inclusive disjunction.

Schröder's views on negation contain much that is interesting and new compared with Boole's views. By negation of class *a* Schröder understands its complementation to unity.[1]

If there are more than two classes, Schröder operates on them according to the rules he formulated.

Rule 1: if the factors of some product include ones among which one is a negation of the other, the product "disappears", i.e., is equal to 0. For example, $abc \cdot ab_{/}\, cd_{/} = 0$, since there exist b and $b_{/}$.

Rule 2: if the members of a sum include at least one that negates another, the entire sum is equal to 1:

$$a + b + c_{/} + a + c + d_{/} = 1$$

Schröder focused considerable attention on analysing the structure of negative propositions. He attached the negative particle to the predicate, i.e., instead of "A is not B" he takes "A is not-B". Thus, the statement "No lion is herbivorous" should, according to Schröder's ideas, be replaced by "All lions are not herbivorous".

[1] *See:* E. Schröder. *Vorlesungen über die Algebra der Logik*, Vol. I, Bronx, N.Y. Chelsea, 1966.

Schröder considers class $a_{/}$ as the negation of class a as very indeterminate. To prove this he gives the following example. The concept of "non-combative" (in the army) embraces: sappers, regimental artisans, and medical personnel, who belong to the army but do not fight.

Relying on De Morgan's laws, Schröder analyses the language of conversational speech. The expression $c \in a_{/} b_{/}$ in speech means that "every c is not-a and (at the same time) not-b". Another expression may be chosen for it: "Every c is neither a nor b". This is a conjunctive proposition which can be exemplified by: "Every fish is not a bird, nor a mammal". Another proposition "No fish is a bird or a mammal" means, in symbolical form $c \in (ab)$, which is equivalent, on the basis of De Morgan's law, to $c \in a_{/} + b_{/}$. The proposition with the negative connective "neither a nor b is c" is presented in the form $a + b \in c_{/}$.

Schröder formulates rules or demands for scientific classification: 1) The set must be identical to the sum of its elements. 2) All elements must be disjunctive, i.e., must exclude one another and in pairs produce 0. 3) There must be one basis for dividing a set into elements. Using negation, Schröder showed how the classified set is divided into elements and subelements.

In logical calculus, taken to the extremes of simplicity, Schröder recognises three basic actions: addition (interpreted as inclusive disjunction), multiplication and negation. He does not, however, consider subtraction to be an unconditionally fulfillable operation.

The author recognises the operation of the subtraction of classes as fully acceptable in logic, but understands it in a completely different way from Boole and Schröder. These two believed that in the difference $a - b$, b must be totally included in a; if, however, $b > a$ or a and b are incompatible, subtraction as an operation cannot be carried out. In contrast to Boole and Schröder, we recognise as possible (i.e., fulfillable) a difference between any two classes a and b, when b may or may not be part of a; as a consequence, we take account of the case of subtraction when classes a and b are empty or universal. This approach is considered in the textbook on pages 73-75.

The best known works by the English logician Stanley Jevons (1835-1882) were *Principles of Science, a Treatise on Logic and Scientific Method* and *Elementary Lessons in Logic, Deductive and Inductive.*

He recognised conjunction, inclusive disjunction, and negation as logical operations, but did not recognise the inverse logical operations of subtraction and division. Jevons designates classes by the letters A, B, C ..., and their complementation to the universal class, designated by 1, or their negation, by the letters *a, b, c...* in italics respectively. He uses 0 for the null (empty) class and replaces the copula by the "equals" sign.

Jevons attaches considerable significance to the principle of substitution, which he formulates thus: provided identity or equivalence exists, then everything that is true about one thing, will also be true about the other. This principle of substitution plays an important role in inference. To designate a relation of equality or identity, Jevons uses the sign "=".

Having designated the positive and negative terms through A and *a*, B and *b*, respectively, Jevons writes down the law of non-contradiction as $Aa = 0$. According to him, the criterion of the falsity of a conclusion is the presence of a contradiction within it, i.e., assertion and negation of one and the same provision, which is written, for example, as the presence of A*a*, B*b*, ABC*a*.

Jevons said that affirmative propositions could be presented in negative form, but he should not have been so categorical in maintaining that there are good grounds for using all sentences in the affirmative form, and that difference (i.e., negative propositions) cannot constitute the basis for inference. Jevons did not deny that assertion and negation, similarity and difference, equality and inequality constitute pairs of equally basic relations, but he maintained that inference was possible only where assertion, similarity or equality, in a word, some type of identity existed or was assumed.

According to the laws of dialectics, identity and difference are two aspects of a single object or process. The reflection of the relations of identity and difference in actual objects of the real world is also expressed in thought, in the forms of conclusions. For this reason, it is wrong and unnecessary to disregard difference, expressed in negative propositions, and to reduce everything merely to identity, expressed in affirmative propositions. The unity of opposites – identity and difference – is inseparable.

Jevons' views on the categorical syllogism with two negative premises are both interesting and original. He asserts that his principle clearly distinguishes cases when inference is correct or incorrect. He gives an example of inference.

> Everything that is not metal is not susceptible to a strong magnetic influence.
> Coal is not metallic.

> Coal is not susceptible to a strong magnetic influence.

Here, from two negative premises, we get a ture negative conclusion. Jevons believes that, when it is possible to substitute one identical thing for another, a conclusion can be drawn from two negative premises.

Jevons made a substantial contribution to the algebra of logic, especially the problem of the negation of classes and negative propositions.

The next stage in the development of mathematical logic was connected with the name of the Russian logician, mathematician and astronomer *Platon Poretsky* (1846-1907). His works[1] do much to generalise and develop the achievements of Boole, Jevons and Schröder.

In an analysis of concepts, Poretsky distinguishes two forms: the form possessing the given property, designated by the letters *a, b, c*..., and the form that does not possess them, designated by $a_,$, $b_,$, $c_,$ and so on.[2] The forms of joint possession or non-possession of several properties are written thus: a, a', b, b' (without any special sign between the letters). The modern intersection of classes Poretsky calls the operation of realisation (multiplication), designating it "·", and the operation of the unification of classes – abstraction (addition),

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[1] P. S. Poretsky, *The Solution of the General Problem of the Theory of Probability Using Mathematical Logic*, Kazan, 1887 and others.

[2] P. S. Poretsky, *On Methods for Solving Logical Equations and the Inverse Method of Mathematical Logic*, Kazan, 1884, p. 111.

designating it "?", i. e., with a question mark. 0 and 1 stand for the null or empty class and the universal class. Poretsky introduces the operation of the negation of classes (negation of *a* is designated by $a_{,}$)–this is a complement to class *a*. For every given *a*, its negation, i. e., $a_{,}$ may be different. This is determined by the chosen universal class. Thus, if Englishmen are taken as 1, i. e., the universe, and the class of performers as *a*, $a_{,}$ stands for Englishmen who are not performers, but if 1 stands for the class of people in general, then $a_{,}$ is the class of people who are not performers, and so on.

It is to Poretsky's credit that he considered logical operations not only with individual logical classes, but also logical equalities. He believed that if two classes consist of one and the same objects, i. e., are of equal extension and can be distinguished only in form, then they are equal. Combining equal classes by the sign "=", we get a logical equality. By equality of logical classes the Russian logician means their total identity, i. e., the equivalence of their logical content, believing that all their differences can only consist in the method of their origination. One example of such an equation is De Morgan's law: $(m + n)_{,} = m_{,}n_{,}$. If classes *a* and *b* are equal, then their negation, i. e., classes $a_{,}$ and $b_{,}$ are also equal. In Poretsky's opinion, the negation of any equality leads to a new equality, identical to the initial one.

The operation of negation on systems of equalities is not suitable, according to Poretsky. Only two logical operations are applicable to a combination of two or more equalities into a single new equality: addition and multiplication of the individual parts of the equations; moreover, if necessary, each individual equality may be replaced in advance by its negation.

In his theory of logic, Poretsky stressed the interconnection between two problems: the deriving of consequences from an invented system of premises and the finding of the premises from which the given logical equation may be obtained as a consequence. Let us look in somewhat more detail at the method for finding all simple consequences from the given premises, which in the theory of logic has become known as the Poretsky-Blake method (it was suggested by the American mathematician Blake[1] on the basis of Poretsky's work).

The simple consequence of the given premises is the disjunction of some letters or their negation, this being a logical consequence of these premises; moreover, it is a consequence that is not absorbed by any other, stronger consequence of the same type. (We say that a is stronger than b if b follows from a, but a does not follow from b).

All simple consequences may be obtained from the given premises by fulfilling the transformation of the following 5 types:

1) Reduce the conjunction of the premises to the conjunction of normal form (CNF). The CNF is a conjunction from the disjunctions of elementary propositions or their negations, equivalent to the given expression (i.e., if there is implication, it must be replaced by disjunction according to the formula $a \to b = \bar{a} \vee b$).

2) Carry out all operations of "exclusion", i.e., the terms of type $a \vee x \vee \bar{x}$ (or $a \cdot x \cdot \bar{x}$) may be excluded, since this term is identically true.

3) Use the laws of revelation, i.e., the formulae $ax \wedge b\bar{x} = ax \wedge b\bar{x} \wedge ab$ or $ax \vee b\bar{x} = ax \vee b\bar{x} \vee ab$.

4) Carry out all "absorptions" on the basis of the laws of absorption: $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$[2].

5) Of all the recurring terms, leave only one (on the basis of the laws of idempotency).

As a result, we get a syllogistic polynomial, which will contain all the simple and only the simple consequences of the given premises. They are more interesting than ordinary logical consequences since they depend on a smaller number of parameters (elementary propositions).

Let us demonstrate this on a specific problem. From the given three premises with the following forms: 1) $q \to \bar{r}$ 2) $p \vee q$ 3) r, derive all different (non-equivalent) forms of simple logical consequences. To solve this problem we shall carry out the following operations:

1. Unite the premises with signs of conjunction and reduce the expression to the CNF:

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[1] A. Blake, *Canonical Expressions in Boolean Algebra*, Chicago, The University of Chicago Libraries, 1938.

[2] The laws of absorption for classes were considered on pages 56-58.

$(q \rightarrow \bar{r}) \wedge (p \vee q) \wedge r = (\bar{q} \vee \bar{r}) \wedge (p \vee q) \wedge r$ or: $\bar{q}\bar{r} \wedge pq \wedge r$.

2. In the CNF obtained, we apply the law of revelation to terms 1 and 3, to obtain $\bar{q}\bar{r} \wedge pq \wedge r = \bar{q}\bar{r} \wedge pq \wedge r \wedge \bar{q}$. Then apply this law again, this time to terms 2 and 4.

$$\bar{q}\bar{r} \wedge pq \wedge r \wedge \bar{q} = \bar{q}\bar{r} \wedge pq \wedge r \wedge \bar{q} \wedge \mathrm{p}.$$

3. Carry out the "absorption" operation. The first term ($\bar{q}\bar{r}$) is absorbed by the fourth ($\bar{q}$), so we discard the first term, and the second term (pq) is absorbed by the fifth (p). As a result we obtain:

$$\bar{q}\bar{r} \wedge pq \wedge r \wedge \bar{q} \wedge p = r \wedge \bar{q} \wedge p.$$

Conclusion: with the given premises, the propositions r and p are true, while the proposition q is false, i.e., if certain events are expressed by the proposition, event r and event p will occur, while event q will not.

Today Poretsky's research still exerts a stimulating impact on the development of algebraic theories of logic.

In the 20th century, mathematical logic has been developed in the works of Charles Pierce and Giuseppe Peano.

The American logician Charles Pierce (1839-1914) made a substantial contribution to the development of algebraic-logical conceptions and was the founder of a new science – semiotics (the general theory of signs). Pierce's works contain a tendency towards a break-down of semiotics into pragmatics (analysing the relation of the sign to its investigator), semantics (explaining the relation of the sign to the object it designates) and syntactics (studying the interrelations between signs).

Pierce writes that something the properties of which are independent of what is thought about them is real. The most general break-down of signs he considered to be the following: icons, indices and symbols. Pierce proposed a classification of signs according to other principles, too.

He suggested building a propositional calculus on only one operation, thereby anticipating the results obtained by M. Sheffer (who also built a propositional calculus on a single operation, that came to be known as

Sheffer's stroke). Pierce suggested negation of inclusive disjunction as the single logical operation.

Pierce's work included *Studies in Logic* and others.

Giuseppe Peano (1858-1932) was an Italian mathematician, whose achievements constituted the transitional link between the algebra of logic in the form it assumed in the works of Boole, Schröder, Poretsky and Pierce, to modern mathematical logic. Peano's chief results were published in a five-volume work: *Formulaire de mathématiques.*[1]

Peano introduced the following symbols still used today: a) "$\in$"–the sign of an element belonging to a class; b) "$\supset$"–the sign of inclusion of one class in another; c) "$\cup$"–the sign of the unification of classes; d) "$\cap$"–the sign designating the operation of the intersection of classes.

Peano made a major contribution to the development of the axiomatic method in his system of five axioms for the arithmetic of natural numbers. Peano builds his entire theory of natural numbers on the basis of these axioms.

At the concluding stage of his scientific work, Peano set out to give a systematic presentation of logic as a special mathematical discipline.

Mathematical logic developed further in many directions, solving a number of problems in the process. These arose from the need for further mastering of classical and non-classical logic, and coping with the difficulties encountered in the substantiation of mathematics.

The following sections of this chapter will give a brief account of these main directions.

§ 2. The Development of Logic in Connection With the Substantiation of Mathematics

The German mathematician and logician Gottlob Frege (1848-1925) attempted to reduce mathematics to

[1] G. Peano, *Formulaire de mathematiques*, Turin, 1901.

logic. To this end, in his first work on mathematical logic *Begriffsschrift*, he defined the set as the extension of a concept and thus was also able to define the number through the extension of a concept. He formulated this definition of a number in his *Grundlagen der Arithmetik*, a book that, at the time, attracted no attention, but later became widely known. Here Frege defines the number belonging to a concept as the extension of this concept. Two concepts are considered to be equinumerous if the sets expressing their extensions can be presented in a univalent correspondence. Thus, for example, the concept of the "vertex of a triangle" is equal in number to that of the "side of a triangle" and each of them has one and the same number 3, which is the extension of the concept of the "vertex of the triangle".

Leibnitz merely outlined a programme for reducing mathematics to logic, but Frege attempted to reduce quite a considerable part of arithmetic to logic, i.e., carried out a certain mathematisation of logic.[1] The symbols he adopted are very cumbersome, so there are few people who have read the whole of his *Grundgesetze der Arithmetik*. Frege himself was not really counting on his work being read. Even so, it played a significant role in the history of the substantiation of mathematics in the first half of the 20th century. In this work, Frege wrote: "In my *Grundlagen der Arithmetik* (1884) I tried to produce arguments to support the idea that arithmetic is a part of logic and must not borrow any principles of proof either from experience or from contemplation. In this book, this will be confirmed by the fact that the simplest laws of arithmetic are derived here only with the help of logical means."[2]

Thus, Frege presumed that he had logically defined the number and precisely listed the logical rules with the help of which new concepts might be defined and theorems proved, and that he had thus also made arithmetic a part of logic. Frege did not suspect, however, that the system he had built not only did not

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[1] *See:* G. Frege. *Grundgesetze der Arithmetik*, Vol. I, Jena, 1893, Vol. II, 1903.

[2] *Ibid.*, Vol. I, p. 1.

constitute a logical substantiation of arithmetic, but was even contradictory. This contradiction in Frege's system was discovered by Bertrand Russell.

In the afterword to *Grundgesetze der Arithmetik*, Frege wrote on this: "There is probably nothing less desirable for the author of a scientific work than the discovery, once it is complete, that one of the foundations of his building was badly shaken (*erschüttert*). I found myself in just such a position when I read the letter sent me by Mr. Bertrand Russell, when this book was due to come out."[1] The contradiction that Russell discovered in Frege's system was Russell's famous paradox of the set of all normal sets (see: pages 199-201 of the textbook).

Frege saw the reason for his failure in his assumption that every concept has an extension in the sense of a constant, strictly fixed set that does not contain any indeterminacy or vagueness. After all, it is precisely through this extension that he determined the basic concept of mathematics, the number.

After Frege, the next attempt to reduce mathematics to logic was made by the eminent English philosopher and logician Bertrand Russell (1872–1970). He also wrote a number of works on history, literature, pedagogics, aesthetics, the natural sciences, sociology, and other fields of knowledge. Russell's works in the sphere of mathematical logic exerted a considerable influence on its development. Together with the English logician and mathematician Alfred Whitehead, Russell developed an original system of symbolic logic in his fundamental three-volume work *Principia Mathematica*.[2] Putting forward the idea of reducing mathematics to logic, Russell believed that, if the hypothesis applied not to a single or several things, but to any object, such conclusions constituted mathematics. In this way, he defined mathematics as a doctrine in which we never know what we are talking about or whether what we are saying is true.

Russell distinguished between pure and applied mathematics. Pure mathematics, in his opinion, is the

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[1] G. Frege, *op. cit.*, Vol. II, p. 253.

[2] Bertrand Russell and A.N. Whitehead, *Principia Mathematica*, Cambridge, 1950.

aggregate of formal conclusions, independent of their content, i. e., a class of statements expressed exclusively in the terms of variables and only logical constants. Not only was Russell quite convinced that he had managed to reduce mathematics to this type of sentences, but also asserted, on this basis, the existence of *a priori* knowledge, believing that "mathematical knowledge needs premises which are not based on the data of sense".[1] Hence it can be seen that Russell separates two interconnected stages of cognition – the sensual and the rational. In mathematics, he discards the first stage of cognition and proceeds immediately to abstract thinking, and this is apriorism, a striving to prove that mathematical truths are truths of reason, in no way connected with experience or the sensual perception of the world.

Russell distinguishes pure mathematics from applied, which consists in the application of formal conclusions to material data.

In order to show that pure mathematics comes down to logic, Russell takes the system of arithmetical axioms formulated by Peano and tries to prove them logically, and to define Peano's three undefined concepts: "zero", "number" and "following" in terms of his own logical system. Russell also considers it possible to express all natural numbers in terms of logic and, consequently, to reduce arithmetic to logic. Since, in his opinion, all pure mathematics can be reduced to arithmetic, then mathematics can be reduced to logic. Russell writes: "Logic has become more mathematical and mathematics has become more logical. The consequence is that it has now become wholly impossible to draw a line between the two; in fact, the two are one. They differ as boy and man: logic is the youth of mathematics and mathematics is the manhood of logic."[2]

In reality, however, mathematics cannot be reduced to just logic. These sciences have different subject-matters. We have already shown the characteristic features inherent in logic as a science (see pages 116-118 of the

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[1] B. Russell, "The Philosophical Importance of Mathematical Logic", *Monist*, Vol. XXII, 1913, No. 4, p. 489.

[2] B. Russell, *Introduction to Mathematical Philosophy*, George Allen and Unwin, London, 1924, p. 194.

textbook). Mathematics has different tasks and functions.

There are two sides to Russell's big three-volume *Principia Mathematica*. The first establishes it as one of the main sources of modern mathematical logic. Everything connected with this side of *Principia Mathematica* was subsequently developed in mathematical logic to the extent that this new sphere of science became particularly important for solving not only the most complex tasks of theoretical mathematics and its substantiation, but also a whole number of problems, important for practice, in computational mathematics and technology.

The other side of the work or, to be precise, not of the work itself, but of the philosophical "generalisations" made by logicists on its basis, includes it among attempts to "prove" the proposition that mathematics comes down to logic. It is this side that contains incorrect conclusions, that have been refuted by the further development of science, which has shown Russell's attempt to be a failure. This is not accidental. The fact is that, in one sense, Russell did not build his system very felicitously, and that no formal "logical system" with precisely listed and effective rules of inference, in which it would be possible to formalise all arithmetic, can actually be built at all. This constitutes the content of the well-known theorem by the Austrian mathematician and logician Kurt Gödel on the incompleteness of formalised arithmetic[1]. From this it immediately follows that the definition of mathematical concepts in terms of "logic" does disclose certain links between these concepts and logic, yet it does not deprive them of their specifically mathematical content. A formalised system only has sense if there is a substantive scientific theory that the given formalised system must serve to systematise.

In their logical analysis, Frege and Russell came, however, to a number of interesting results concerning the concepts: "object", "name", "meaning", "sense",

[1] K. Gödel, "Über formal unentscheidbare Sätze der *Principia Mathematica* und verwandter Systeme ", *Preussische Akademie der Wissenschaften.* Sitzungsberichte der Preussische Akademie der Wissenschaft, Vol. 38, Berlin, 1930.

"function", "relation" and others. Of particular significance was Russell's theory of types (simple and extended), the goal of which is to resolve the paradoxes in the theory of sets. There is a rational core in Russell's extended theory of types – the fact that it is a constructive theory.

§ 3. Intuitionistic Logic

Intuitionistic logic was developed in connection with the growth of intuitionistic mathematics. The intuitionistic school was founded in 1907 by the Dutch mathematician and logician L. Brouwer (1881-1966)[1], but some of its ideas had been put forward previously.

Intuitionism is a philosophical trend in mathematics and logic, rejecting the use of the abstraction of actual infinity and logic as a science preceding mathematics, and regarding intuitive clarity and cogency ("intuition") as the ultimate basis of mathematics and logic. The intuitionists build their mathematics with the help of finite means on the basis of a system of natural numbers that is considered to be known intuitively. Intuitionism includes two aspects – the philosophical and the mathematical.

The mathematical content of intuitionism is set out in a number of works by mathematicians. The leading Soviet constructivist mathematicians note the positive significance of some of the intuitionists' mathematical ideas.

In general, constructive mathematics differs substantially from intuitionistic mathematics. Yet the Soviet constructivist mathematician A. Markov writes that the constructive trend has points of contact with so-called intuitionistic mathematics. The constructivists share the intuitionists' understanding of disjunction, thus recognizing Brouwer's critique of the law of the excluded middle. At the same time, the constructivists consider

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[1] L.E.J. Brouwer, "Intuitionism and Formalism", *Bulletin of American Mathematical Society*, Vol. 20, 1913; "The Effect of Intuitionism on Classical Algebra of Logic", *Proceedings of the Royal Irish Academy*, 1955, Vol. 57, pp. 113–116.

the methodological principles of intuitionism to be unacceptable.

Thus, the mathematical and the philosophical aspects of intuitionism should be clearly distinguished. Whereas the former has a rational part (in this connection, it is preferable to speak of intuitionistic mathematics or intuitionistic logic, rather than intuitionism), the latter aspect of intuitionism – its methodological, idealistic, philosophical foundations – is totally unacceptable.

Brouwer believed that pure mathematics was a free creation of the human reason and had nothing to do with experimental facts. The intuitionists saw intuition as the only source of mathematics and "intuitive clarity" as the criterion for judging the acceptability of mathematical concepts and conclusions. Yet, the intuitionist Arend Heyting was forced to admit that the concept of intuitive clarity in mathematics is not in itself intuitively clear; a declining scale of degrees of obviousness could even be constructed.

The basis on which mathematics emerged was ultimately not some "intuitive clarity", being the product of human consciousness, but a reflection of spatial forms and quantitative relations in the real world. Heyting, like Brouwer, is a subjective idealist in gnoseology. He maintains that mathematical thought characteristically does not reflect a truth about the outside world, but is connected exclusively with mental constructions.[1]

Back in 1936, the Soviet mathematician A. Kolmogorov criticised the subjective idealistic foundations of intuitionism, declaring the intuitionists to be mistaken in claiming that mathematical objects were a product of the constructive activities of our mind, for mathematical objects are abstractions of really existing forms, independent of the human intellect. The intuitionists do not recognise human practice and experience as a source for forming mathematical concepts, methods of mathematical constructions and methods of proof.

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[1] *See:* A. Heyting. "Die formalen Regeln der intuitionistischen Logik", Sitzungsberichte der preussischen Akademie der Wissenschaft, Berlin, 1930; *Intuitionism. An Introduction*, Amsterdam (North Holland Publishing Co.), 1956.

The *specifics of intuitionistic logic* derive from the characteristic features of intuitionist mathematics.

In modern classical mathematics, indirect proofs are often used, but it is virtually impossible to do so in intuitionistic mathematics and logic, since the law of the excluded middle and that of $\bar{\bar{a}} \to a$, which participate in indirect proofs, are not recognised here.

The law of the excluded middle for infinite sets does not apply in intuitionist logic because $p \vee 7p$* requires a general method for solving any problem or, more clearly, one that, for the arbitrary statement p, would allow either p or the negation of p to be proved. Heyting believes that, since the intuitionists do not possess such a method, they have no right to assert the principle of the excluded middle. Let us give an example of this, taking the assertion: "Any whole number greater than unity is either a prime, or the sum of two primes or the sum of three primes". We do not know whether this is true, though it does hold for the finite number of cases considered. Does a number exist that does not satisfy this requirement? We cannot indicate such a number, nor derive a contradiction from assuming its existence.

This is the famous Goldbach problem, which was posed by mathematician Christian Goldbach in 1742, and defied solution for 200 years. Goldbach proposed that any whole number greater than or equal to six could be presented in the form of the sum of three prime numbers. For odd numbers it was solved positively only in 1937, by the Soviet mathematician Academician Ivan Vinogradov: all sufficiently large odd numbers can be represented as the sum of three prime numbers. This was one of the greatest discoveries in modern mathematics. The law of non-contradiction is, however, considered to be unrestrictedly applicable by representatives of both intuitionistic and constructive logic.

Brouwer was the first to give the outline of the new logic. His ideas were formalised by Heyting who, in 1930, built his intuitionistic propositional calculus using implication, conjunction, disjunction and negation on the basis of eleven axioms and two rules of inference –

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* 7 is a sign of negation.

modus ponens and the rule of substitution. Heyting asserts that, although the main differences between classical and intuitionistic logic concern the properties of negation, these logics certainly do not coincide even in formulas without negation. He distinguishes mathematical from actual negation: the former is expressed as a constructive fulfilment of a specific action, while the latter concerns the non-fulfilment of an action (while "non-fulfilment" of something cannot be a constructive action). Intuitionistic logic is concerned only with mathematical statements and only with mathematical negation, which is determined through the concept of contradiction, while the concept of contradiction is considered by the intuitionists to be primary, expressed or presented in the form $1 = 2$. Actual negation is not connected with the concept of contradiction.

The Soviet philosopher A. Nikiforov, in his article "Modification of Tabular Constructions for Classical and Intuitionistic Propositional Logic", formulated tabular proof methods relying on the concept of the "inference tree".

The Soviet philosophers K. Sukhanov, M. Panov and others are studying problems of intuitionistic logic.

§ 4. Constructive Logics

Constructive logic, in contrast to classical logic, derives from constructive mathematics. Constructive mathematics can be described, in short, as the abstract speculative science of constructive processes and our ability to accomplish them. As a result of a constructive process, a constructive object emerges, i. e., one that is given by an effective (precise and quite clear) means of construction (algorithm).

The constructive trend (in mathematics and logic) studies only constructive objects and brings them within the bounds of abstraction of potential realisability, i. e., it ignores the practical restrictions on our possibilities for constructions in space, time and material.

There are points of contact between the constructive logic ideas of Soviet researchers and some of those of intuitionistic logic (for example, in the understanding of disjunction, and in the rejection of the law of the excluded middle).

There are also, however, fundamental differences between constructive and intuitionistic logic.

1. *Different objects of study.* Constructive logic, which is the logic of constructive mathematics, is based on abstraction of potential realisability, and it studies only constructive objects (words in a given alphabet).

Intuitionistic logic, which is the logic of intuitionistic mathematics, is based on the idea of "freely established sequence" i.e., not built according to an algorithm, which the intuitionists consider to be intuitively clear.

2. *Intuitionistic mathematics and logic are substantiated* with the help of idealistically interpreted intuition, while the substantiation of constructive mathematics and logic is based on the scientific mathematical concept of the algorithm (for example, A. Markov's normal algorithm) or the equivalent of the recursive function.

3. *Different methodological principles.* The methodological basis of the constructive trend in mathematics is considered by Soviet scholars to be the provisions of dialectical materialism, which make practice the criterion of the truth of cognition (including scientific). This provision retains its force for such sciences as logic and mathematics, too, although here practice is included in the process of cognition only in a mediated way, in the final count.

The intuitionists, however, remain within the framework of subjective idealistic philosophy, and see not human practice as the source of the formation of mathematical concepts and methods but primary "intuition", and regard "intuitive clarity" as the criterion of truth in mathematics.

4. *Different interpretations**. Kolmogorov interpreted intuitionistic logic as the calculus of problems. A. Markov interpreted logical links of constructive logic as applied to potentially realisable constructive processes (actions).

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* Interpretation (in mathematical logic) means the extension of the initial provisions of some formal system to some substantive system, the initial provisions of which are defined independently of the formal system.

The intuitionistic logic of Brouwer and Heyting is interpreted by its authors as sentential (propositional) calculus, moreover the sphere of propositions for them is confined to mathematical sentences.

5. *Differences in certain logical means.*

Soviet representatives of strictly constructive logic recognise the following principle: if some algorithmic process exists and it has been proved that it does not continue infinitely, then, consequently, the process will end. Some representatives of constructive logic prove this in a more precise form.

The representatives of intuitionistic logic do not recognise this principle.

The constructive propositional calculi of V.I. Glivenko and A.N. Kolmogorov

The first constructive logicians were the Soviet mathematicians A. Kolmogorov (1903-1987) and V. Glivenko (1897-1940). The first calculus not containing the law of the excluded middle was proposed in 1925 by Kolmogorov in connection with his critique of Brouwer's conception, and was later developed by Glivenko. Subsequently, Heyting's calculus was published, and Kolmogorov interpreted this as a calculus of problems that engendered an interpretation of calculi not using the law of the excluded middle. This, in turn, provided the basis for further genuinely scientific research into such calculi.

With the help of the introduction of the concepts of "pseudotruth" (double negation of propositions) and "pseudomathematics" ("mathematics of pseudotruth"), Kolmogorov proved that any conclusion obtained with the help of the law of the excluded middle is true if each of the propositions included in its formulation is replaced by a proposition asserting its double negation. Thus he showed that, in the "mathematics of pseudotruth" it is lawful to apply the principle of the excluded middle.

Kolmogorov distinguishes two logics of propositions – the universal and the particular. The difference between them lies in a single axiom $\overline{\overline{A}} \rightarrow A$, which is included only among the axioms of particular logic. The dialectics of the correlation between the contents and spheres of application of these logics are interesting: the content of the particular logic of propositions is richer than that of the general, since particular logic includes the additional axiom $\bar{\bar{A}} \rightarrow A$, but its sphere of application is narrower. All the formulas of the traditional logic of propositions can be deduced from the system of particular logic.

What then, is the sphere of application of the particular logic of propositions? All its formulas are true for statements of type $A^{\cdot}$, including for all finite and all negative statements, i.e., its sphere of application coincides with that of the formula of double negation $\bar{\bar{A}} \rightarrow A$. (The symbols $A^{\cdot}$, $B^{\cdot}$... designate arbitrary statements, for which the statement itself follows from the double negation).

The constructive logic of A. A. Markov

The problem of the constructive understanding of logical connectives, in particular negation and implication, requires that special, precise, formal languages be used in logic. The constructive mathematical logic of the Soviet mathematician A.A. Markov (1903-1979) is based on the idea of a ranked structure of formal languages. First, the formal language L_0 is introduced, in which sentences are expressed by formulas according to certain rules. It includes a definition of the sense of the expression of this language, i.e., semantics. The rules of inference always allow true sentences to be obtained by proceeding from true sentences.

Constructive mathematics formulates the theorems of existence, asserting that there exists an object that satisfies certain demands. In this it is assumed that the construction of this object is potentially realisable, i.e., that we possess a means for creating it. This constructive understanding of existential propositions differs from the classical one. The interpretation of disjunction is also different in constructive mathematics and logic, being understood as the realisability of indicating its true element. "Realisability" means the potential realisability of the constructive process giving, as a result, one of the elements of the disjunction, which must be true. The classical understanding of disjunction does not, however, presume that its true element can be found.

A new understanding of logical connectives requires a new logic. We consider Markov's assertion concerning the non-uniqueness of logic to be true and very profound: "There is nothing surprising, of course, in the actual idea of the non-uniqueness of logic. In fact, why should all our arguments, about everything whatsoever, be governed by the same laws? There are no grounds for this. It would be surprising, on the contrary, if logic were unique".[1]

Markov introduces into constructive mathematical logic the concept of the "solvable proposition" and the related one of "direct negation". Markov's logic contains another type of negation, too – intensified negation, in relation to so-called semi-solvable propositions.

Apart from material and strong implication, the truth of the premises and conclusion being of primary concern in establishing the truth of these, Markov introduces deductive implication, defined according to another principle. The deductive implication "if A, then B" expresses the possibility of deducing B from A according to fixed rules, each of which, when applied to true formulas, gives true formulas. Any statement deduced from a true statement is true.

Through deductive implication, Markov defines reductive negation (*reductio ad absurdum*). The reductive negation of a statement A (formulated in the given language) is understood as the deductive implication "if A, then L", where L designates an absurdity. This definition of negation corresponds to the usual practice of mathematical argument: the mathematician denies anything he can reduce to the absurd. There is no need to fathom the sense of a statement in order to establish the truth of its reductive negation. A statement for which the truth of the reductive negation is established cannot itself be true.

These three different understandings of negation do not conflict with one another; they are co-ordinated and this fact, according to Markov, provides an opportunity to combine them.

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[1] A. Markov, "On the Logic of Constructive Mathematics", *Bulletin of Moscow University* (in Russian).

The following circumstance is indicative: Markov builds his constructive logical systems for substantiating constructive mathematics in such a way that be obtains not just one complete system, but a whole hierarchy of systems. These are the system of languages L_0, L_1, L_2, L_3, L_4, L_5, ..., L_n (where n is a natural number) and the universal language $L_{(\omega)}$ which includes them all; after $L_{(\omega)}$, the language $L_{(\omega|)}$ is built.

We are thus inclined to think that the developing constructive logic and mathematics cannot fit into a single formal calculus. For this purpose a system is required that consists of a whole hierarchy of systems, in which there will be a hierarchy of negations.

The problems of constructive logic and the theory of algorithms are also being studied by the Soviet mathematician N. M. Nagorny and others.

§ 5. Multi-Valued Logics

In multi-valued logics, the number of values of the truth of the propositional arguments and functions can be any finite (more than two) or even infinite number. In this section we shall use the so-called Polish symbols of Lukasiewicz and the usual ones applied in two-valued or binary logic: negation is designated by N_x or $\bar{x}$, and conjunction by Kxy or $x \wedge y$, inclusive disjunction by Axy or $x \vee y$, material implication by Cxy or $x \rightarrow y$. The value of the function from the argument a is written thus: $[a]$. A formula that, given any values of its variables, assumes the indicated value is called a *tautology* (or universal, or a law of logic or identically true). As a rule, this is the value "truth" (in the systems considered, "thuth" is usually designated by the figure 1).

The development of multi-valued logics confirms the idea that truth is always concrete and concrete scientific knowledge relative: that which is identically true in one logical system, cannot be so in another.

Lukasiewicz's three-valued or ternary system

The three-valued propositional logic (the logic of statements) was constructed in 1920 by the Polish

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mathematician and logician J. Lukasiewicz (1878-1956). In it "truth" is designated by 1, "null" by 0 and "neutral" by 1/2. The chief functions are taken to be negation (*Nx*) and implication (*Cxy*); the derivatives are conjunction (*Kxy*) and disjunction (*Axy*). The tautology assumes the value 1.

Negation and implication are defined respectively by matrices (tables) thus:

Lukasiewicz's negation

x	Nx
1	0
$^1/_2$	$^1/_2$
0	1

Lukasiewicz's implication

$x \backslash y$	1	$^1/_2$	0
1	1	$^1/_2$	0
$^1/_2$	1	1	$^1/_2$
0	1	1	1

$$[Nx] = 1 - [x].$$

Conjunction is defined as the minimum value of the arguments: $[Kxy] = \min([x], [y])$; disjunction as the maximum values of x and y: $[Axy] = \max([x], [y])$.

The table for Lukasiewicz's implication, expressed in the form $x \to y$ is used as follows: the first column on the left contains the values of x, and on the top are written the values of y. Let us take, for example, $[x] = 1/2$ (i. e., the value of x is equal to 1/2), and $[y] = 0$. We obtain $1/2 \to 0$. At the intersection we get the result 1/2.

If the formula includes one variable, for example, the formula $a \vee \bar{a}$, then the table of truth for this formula, including all possible combinations of truth or falsity or indeterminacy of its variable in the table, will consist of $3^1 = 3$ lines; for two variables the table will have $3^2 = 9$ lines; for three variables we have $3^3 = 27$ lines; for n variables there will be 3^n lines.

Let us show how the formula $a \vee \bar{a}$ (the law of the excluded middle) and $\overline{a \wedge \bar{a}}$ (the law of non-contradiction), containing one variable, i.e., a, are proved. The table will have only $3^1 = 3$ lines.

a	$\bar{a}$	$a \vee \bar{a}$	$a \wedge \bar{a}$	$\overline{a \wedge \bar{a}}$
1	0	1	0	1
$1/2$	$1/2$	$1/2$	$1/2$	$1/2$
0	1	1	0	1

To prove the formula $a \vee \bar{a}$ we use our knowledge that disjunction is taken at the maximum. In the third column, corresponding to $a \vee \bar{a}$ we see that, together with the values 1, there is the value 1/2. Consequently, this formula is not a law of logic. Columns 4 and 5 are constructed similarly, but they observe the condition that conjunction is taken for the minimum of values. The formula $\overline{a \wedge \bar{a}}$ is not a law of logic either.

Now let us consider whether the formula containing two variables x and y is a law of logic. In the table there will be $3^2 = 9$ lines. The distribution of the values of truth for x and y is shown in the first and second columns.

The formula is thus: $(x \rightarrow (\bar{y} \wedge y)) \rightarrow \bar{x}$

x	y	$\bar{x}$	$\bar{y}$	$\bar{y} \wedge y$	$x \rightarrow (\bar{y} \wedge y)$	$(x \rightarrow (\bar{y} \wedge y)) \rightarrow \bar{x}$
1	1	0	0	0	0	1
1	$1/2$	0	$1/2$	$1/2$	$1/2$	$1/2$
1	0	0	1	0	0	1
$1/2$	1	$1/2$	0	0	$1/2$	1
$1/2$	$1/2$	$1/2$	$1/2$	$1/2$	1	$1/2$
$1/2$	0	$1/2$	1	0	$1/2$	1
0	1	1	0	0	1	1
0	$1/2$	1	$1/2$	$1/2$	1	1
0	0	1	1	0	1	1

Conclusion: since, in the last column, the value of indeterminacy (i. e., 1/2) is encountered twice, the given formula is not a law of logic.

On the basis of these definitions of negation, conjunction and disjunction, the law of non-contradiction and the law of the excluded middle of two-valued logic will not be tautologies in Lukasiewicz's system. Neither are the negations of the laws of non-contradiction and of the excluded middle tautologies in Lukasiewicz's system. Lukasiewicz's logic is not, therefore, a negation of two-valued logic. In his logic, the rule of the removal of double negation, all four of De Morgan's rules and the rule of contraposition: $a \to b \equiv \bar{b} \to \bar{a}$ are tautologies. The rule of *reductio ad absurdum* of two-valued logic: $(x \to \bar{x}) \to \bar{x}$ and $(x \to (\bar{y} \wedge y)) \to \bar{x}$ (i. e., if a contradiction derives from x, negation of x follows from this) are not tautologies. This may be proved taking $[x] = 1/2$ and $[y] = 1/2$.

In Lukasiewicz's system, certain formulas structurally expressing the correct deductive inferences of traditional logic, formalised by means of the algebra of logic, are not tautologies. These are *modus tollens*, the simple destructive dilemma, and the formulas of the disjunctive categorical syllogism with inclusive disjunction.

All the tautologies of Lukasiewicz's logic are tautologies in two-valued logic, for, if the value 1/2 is discarded, the definition of the functions of conjunction, disjunction, implication and negation coincide respectively in Lukasiewicz's logic and two-valued logic. Since, in the former, however, there is a third value of truth – 1/2, not all tautologies of two-valued logic are tautologies in Lukasiewicz's.

Heyting's three-valued system

In two-valued logic, it follows from the law of the excluded middle that: 1) $\bar{\bar{x}} \to x$; 2) $x \to \bar{\bar{x}}$. Proceeding from the assertion that only the second of these is true, the Dutch logician and mathematician A. Heyting (1898-1980) elaborated a three-valued system of propositional logic. In this logical system, implication and negation differ from Lukasiewicz's definitions of these operations only in one case. "True" is designated by 1, "false" by 0, "indeterminate" by 1/2. Tautology assumes the value of 1.

Heyting's negation

x	Nx
1	0
$^1/_2$	0
0	1

Heyting's implication

x \ y	1	$^1/_2$	0
1	1	$^1/_2$	0
$^1/_2$	1	1	0
0	1	1	1

Conjunction and disjunction are defined in the usual way as the minimum and maximum values of the arguments.

Taking only the values of the functions 1 and 0 into account, the matrices of the two-valued logic can be obtained from Heyting's matrices. In this three-valued logic, the law of non-contradiction is a tautology, but neither the law of the excluded middle nor its negation are tautologies. Both the correct modi of the conditional categorical syllogism, the formula $(x \to y) \to (\bar{y} \to \bar{x})$, De Morgan's rules and the law of the excluded fourth: $(x \vee \bar{x} \vee \bar{\bar{x}})$ are tautologies.

Although, in comparison with Lukasiewicz's logic, Heyting introduced certain small changes into the matrices of negation and implication in his system, the results proved significant: in Heyting's system many formulas of the classical two-valued propositional calculus are tautologies.

Post's m-valued system (Pm)[1]

The system developed by the American logician and mathematician Emil Post (1897-1954) is a generalisation of the two-valued logic, for given $m = 2$ as the particular case, we obtain the two-valued logic. The values of truth are $1, 2, \dots, m$ (with $m \geqslant 2$), where m is the final number. The formula that always assumes the numerical value lying between 1 and $m - 1$ inclusively, is a tautology.

Post introduces two types of negation (N^1x and N^2x), called cyclical and symmetrical negations respectively. They are defined by means of matrices and equations.

The first negation is defined by two equations:

1. $[N^1x] = [x] + 1$ given $[x] \leqslant m - 1$.
2. $[N^1n] = 1$.

The second negation is defined by a single equation:

$$[N^2x] = m - [x] + 1.$$

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[1] *See:* Post E. L. "Introduction to a General Theory of Elementary Propositions". In: *American Journal of Mathematics*, 1921, Vol. 43, No. 3, pp. 163–185.

x	N^1x	N^2x
1	2	m
2	3	$m-1$
3	4	$m-2$
4	5	$m-3$
.	.	.
.	.	.
.	.	.
$m-1$	m	2
m	1	1

One characteristic feature of Post's two negations is that, given $m = 2$, they coincide with each other and with the negation of two-valued logic, which confirms the thesis that Post's multi-valued system is a generalisation of the two-valued logic.

Conjunction and disjunction are defined respectively as the maximum and minimum values of the arguments. Given these definitions of negation, conjunction and disjunction, it turns out that, with values of x greater than 2, the laws of non-contradiction and of the excluded middle, as well as the negation of these laws, are not tautologies.

If the values of truth are 1, 2, 3, then the three-valued logic, i.e., P_3 separates out from Post's multi-valued system. Similarly, given values of truth 1, 2, 3, and 4, a four-valued logic P_4 is obtained, and so on.

The three-valued system P_3 has the following form:

p q

p	$\sim \cap_3 p$	$\approx \cap_3 p$
1	2	3
2	3	2
3	1	1
Explanation	First negation	Second negation

$p \setminus q$	$p \cdot_3 q$			$p \vee_3 q$			$p \supset_3 q$			$p \equiv_3 q$		
	1	2	3	1	2	3	1	2	3	1	2	3
1	1	2	3	1	1	1	1	2	3	1	2	3
2	2	2	3	1	2	2	1	2	2	2	2	2
3	3	3	3	1	2	3	1	1	1	3	2	1
Explanation	$\max(p, q)$			$\min(p, q)$			$(\approx \cap_3 p) \vee \cap_3 q$			$(p \supset_3 q) \wedge \cap (q \supset_3 p)$		

In these tables, the designations introduced by Post given $m = 3$ are used: the first negation is designated by $(\sim_3 p)$ and the second by $(\approx_3 p)$, conjunction by $(p \cdot_3 q)$, disjunction by $(p \vee_3 q)$, implication by $(p \supset_3 q)$ and equivalence by $(p \equiv_3 q)$.

If only 1 "true" and 3 "false" are taken as the values of truth, tables for negation, conjunction, disjunction, implication and equivalence of the two-valued logic separate out from the tables of Post's P_3 system.

In the P_3 system, tautology assumes the value 1; the law of the excluded middle is not a tautology for either the first or the second negation in Post's system, but the law of the excluded fourth is a tautology for the first negation.

Reichenbach's three-valued system[1]

The apparatus of multi-valued logics is finding increasing application in various sciences. Let us analyse the application of the apparatus of the three-valued logic developed by the German philosopher and logician H. Reichenbach (1891-1953) to quantum mechanics.

The majority of the operations of this system were already introduced by Post, but Reichenbach introduced new ones for the purpose of applying his system to quantum mechanics. Post introduced two negations – the first and the second. In Reichenbach's system they are called cyclical negation and diametrical negation. In addition to these, Reichenbach also introduced complete negation. In the latter's system there are standard implication $(\supset)$ and standard equivalence $(\equiv)$, and other operations are introduced: alternative implication $(\rightarrow)$, quasi-implication $(\ni)$ and alternative equivalence $(\equiv)$. The sign "$\cdot$" is used to designate conjunction and "$\vee$" – disjunction.

The table is for the three types of Reichenbach's negation. Designations: $\sim A$ – cyclical negation; $-A$ – diametrical negation; $\bar{A}$ – complete negation.

[1] *See:* H. Reichenbach, *Philosophic Foundations of Quantum Mechanics*, University of California Press, Berkeley and Los Angeles, 1946, § 32.

Reichenbach designated "true" as 1, "indeterminate" as 2, and "false" as 3. Tautology assumes the value 1.

A	$\sim A$	$-A$	$\bar{A}$
1	2	3	2
2	3	2	1
3	1	1	1

Reichenbach's other functions are defined by matrices thus:

A	B	$A \cdot B$	$A \vee B$	$A \supset B$	$A \rightarrow B$	$A \ni B$	$A \equiv B$	$A \equiv B$
1	1	1	1	1	1	1	1	1
1	2	2	1	2	3	2	2	3
1	3	3	1	3	3	3	3	3
2	1	2	1	1	1	2	2	3
2	2	2	2	1	1	2	1	1
2	3	3	2	3	1	2	2	3
3	1	3	1	1	1	2	3	3
3	2	3	2	1	1	2	2	3
3	3	3	3	1	1	2	1	1

Let us note a number of properties inherent in negation in Reichenbach's system.

For cyclical negation, the law of the detachment of ternary negation is true: $\sim \sim \sim A \equiv A$, i. e., as a result of ternary negation of A we return to the initial value of A. For cyclical negation, the laws of non-contradiction and the excluded middle, and De Morgan' laws of two-valued logic are not tautologies, but the law of the excluded fourth $A \vee \sim A \vee \sim \sim A$ is a tautology.

For diametrical negation, the rule of the detachment of binary negation is retained: $- - A \equiv A$. Neither the laws of non-contradiction and the excluded middle themselves, nor their negations are tautologies in diametrical negation.

For complete negation, the law of non-contradiction, the pseudo-law of the excluded middle, the law of the excluded fourth, De Morgan's laws, and the law $\bar{A} \equiv \bar{\bar{\bar{A}}}$ all prove to be tautologies.

Having considered the three forms of negation in their interconnections, Reichenbach showed that the following relationship exists between cyclical and complete negation:

$$\bar{A} \equiv \sim A \vee \sim \sim A. \qquad (1)$$

It has already been noted that, for cyclical negation, the law of the excluded fourth: $A \vee \sim A \vee \sim \sim A$, is a tautology. Its last two terms can be replaced on the basis of equation (1) by $\bar{A}$ and thus the formula $A \vee \bar{A}$ obtained for complete negation. Reichenbach called this the "pseudo-law of the excluded middle", for it does not have the properties of the law of the excluded middle in two-valued logic. The reason is that complete negation does not have the properties of ordinary negation: it does not allow us to determine the value of the truth of A if we know that $\bar{A}$ is true. It follows from the table defining complete negation that, if $\bar{A}$ is true, then A may be either false or indeterminate.

A	$\bar{A}$
t	i
i	t
f	t

The consequence of the ambivalence of complete negation is that the inverse operation, i.e., that leading from $\bar{A}$ to A, cannot be defined.

The interconnection between the three types of negation is expressed in the fact that the law of non-contradiction is retained in the following three forms:

1) $\overline{A \cdot \bar{A}}$; 2) $\overline{A \cdot \sim A}$; 3) $\overline{A \cdot - A}$.

Reichenbach built his three-valued system for describing the phenomena of quantum mechanics. In his opinion, it is lawful to speak of the truth or falsity of propositions only when it is possible to test them. If it is not possible either to confirm the truth of a proposition (i.e., verify it) nor to refute by testing (falsify), then such a statement must be assessed by a third value – indeterminate. Such statements include those on unobserved objects in the microworld.

Reichenbach himself writes thus concerning the values of the three-valued logic for quantum mechanics: "The introduction of a third truth value does not make all statements of quantum mechanics three-valued. As pointed out above, the frame of three-valued logic is wide enough to include a class of true-false formulas. When we wish to incorporate all quantum mechanical statements into three-valued logic, it will be the leading idea to put into the true-false class those statements which we call quantum mechanical laws".[1]

The infinite-valued logic as a generalisation of Post's multi-valued system (Getmanova's system)

Proceeding from Post's P_m system, an infinite-valued system $G_{\chi 0}$ can be constructed that will assume the following form. The values of truth are 1 (true), 0 (false) and all fractions in the interval from 1 to 0, constructed in the form $(1/2)^k$ and the form $(1/2)^k \cdot (2^k - 1)$, where k is an integer exponent. In other words, the values of truth in the system $G_{\chi 0}$ are 1, 1/2, 1/4, 3/4, 1/8, 7/8, 1/16, 15/16, ..., $(1/2)^k$, $(1/2)^k \cdot (2^k - 1)$, ..., 0.

The operations: negation, disjunction, conjunction, implication and equivalence in $G_{\chi 0}$ are defined by the following equations:

1) Negation: $[\approx_{\chi 0} p] = 1 - [p]$.
2) Disjunction: $[p \vee_{\chi 0} q] = \max([p], [q])$.
3) Conjunction: $[p \wedge_{\chi 0} q] = \min([p], [q])$.
4) Implication: $[p \supset_{\chi 0} q] = [\approx_{\chi 0} p \vee_{\chi 0} q]$.
5) Equivalence: $[p \equiv_{\chi 0} q] = [(p \supset_{\chi 0} q) \wedge_{\chi 0} (q \supset_{\chi 0} p)]$.

Negation in the system $G_{\chi 0}$ is a generalisation of the second (symmetrical) negation of Post's m-valued logic. It is by means of this negation that conjunction, implication and equivalence are built in the $G_{\chi 0}$ system. This system, built by the proposed method, has a multitude of tautologies and tautology assumes the value 1 within it. For example, the formula according to which the negation of p repeated twice, gives the initial

[1] H. Reichenbach, *Philosophic Foundations of Quantum Mechanics*, p. 160.

value of p: $\bar{\sim}_{\chi 0}(\bar{\sim}_{\chi 0}p) \equiv_{\chi 0} p$ is a tautology. Also tautologies in $G_{\chi 0}$ will be the four De Morgan's laws, for example:

1) $\bar{\sim}_{\chi 0}(a \vee_{\chi 0} b) \equiv_{\chi 0} (\bar{\sim}_{\chi 0} a \wedge_{\chi 0} \bar{\sim}_{\chi 0} b)$;
2) $\bar{\sim}_{\chi 0}(a \wedge_{\chi 0} \bar{\sim}_{\chi 0} b) \equiv_{\chi 0} a \vee_{\chi 0} b$.

The tautologies in $G_{\chi 0}$ are also tautologies in the two-valued logic, for the infinite-valued system $G_{\chi 0}$ is a generalisation of Post's P_m system, while the latter is a generalisation of the two-valued logic.

To verify the construction of $G_{\chi 0}$, a system G_3 was constructed by the method we have suggested, on the basis of the $G_{\chi 0}$ system, taking 1, 1/2, and 0 as truth values. The G_3 system coincides with Post's P_3 system. A four-valued system G_4, with values of the truth of the arguments 1, 1/2, 1/4, 0 and those of the truth of the functions: 1, 1/2, 3/4, 0, likewise separates out from the $G_{\chi 0}$ system. Negation, conjunction, disjunction, implication, and equivalence in G_4 are defined by the following tables.

p	$\bar{\sim}_4 p$
1	0
1/2	1/2
1/4	3/4
0	1

p	q	$p \supset_4 q$	$q \supset_4 p$	$p \equiv_4 q$
1	1	1	1	1
1	1/2	1/2	1	1/2
1	1/4	1/4	1	1/4
1	0	0	1	0
1/2	1	1	1/2	1/2
1/2	1/2	1/2	1/2	1/2
1/2	1/4	1/2	3/4	1/2
1/2	0	1/2	1	1/2
1/4	1	1	1/4	1/4
1/4	1/2	3/4	1/2	1/2
1/4	1/4	3/4	3/4	3/4
1/4	0	3/4	1	3/4
0	1	1	0	0
0	1/2	1	1/2	1/2
0	1/4	1	3/4	3/4
0	0	1	1	1

The four-valued system G_4 contains the classical two-valued logic [given values of truth 1 ("true") and

0 ("false")], as well as Post's P_3 system (with values of truth 1, 1/2, 0).

Similarly, a system G_5, and systems G_6, G_7, G_8 and so on separate out from $G_{\chi o}$.

	$p \wedge_4 q$				$p \vee_4 q$				$p \supset_4 q$				$q \supset_4 p$				$p \equiv_4 q$			
p \ q	1	$\frac{1}{2}$	$\frac{1}{4}$	0	1	$\frac{1}{2}$	$\frac{1}{4}$	0	1	$\frac{1}{2}$	$\frac{1}{4}$	0	1	$\frac{1}{2}$	$\frac{1}{4}$	0	1	$\frac{1}{2}$	$\frac{1}{4}$	0
1	1	$\frac{1}{2}$	$\frac{1}{4}$	0	1	1	1	1	1	$\frac{1}{2}$	$\frac{1}{4}$	0	1	1	1	1	1	$\frac{1}{2}$	$\frac{1}{4}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	0	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	1	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$
0	0	0	0	0	1	$\frac{1}{2}$	$\frac{1}{4}$	0	1	1	1	1	0	$\frac{1}{2}$	$\frac{3}{4}$	1	0	$\frac{1}{2}$	$\frac{3}{4}$	1
Explanation	min([p], [q])				max([p], [q])				[$\approx_4 p \vee_4 q$]				[$\approx_4 p \vee_4 q$]				[($p \supset_4 q$) $\wedge_4$ ($q \supset_4 p$)]			

On interpretation of the $G_{\chi o}$ system

In the $G_{\chi o}$, system between the extreme values of truth 1 ("true") and 0 ("false"), lie an infinite number of values of truth: 1/2, 1/4, 3/4, 1/8, 7/8 and so on. The process of cognition is carried out in such a way that we proceed from ignorance to knowledge, from incomplete, imprecise knowledge to fuller and more precise knowledge, from relative to absolute truth. Absolute truth (in the narrow sense) is formed out of the infinite sum of relative truths. If we attach the semantic sense of absolute truth to the value of truth equal to 1, and to the value 0 that of a falsehood (error, ignorance), the intermediate values of truth will reflect the process of attaining absolute truth as an infinite process, composed of cognition of relative truths, the values of which, in the $G_{\chi o}$ system, are 1/2, 1/4, 3/4, 1/8, 7/8, ... and so on. The closer the value of the truth of the variables (expressing propositions) are to 1, the closer we approach absolute truth. Thus, the process of cognition is accomplished from ignorance to knowledge, from phenomena to essence, from essence of the first order to essence of the second order, and so on. This infinite process of cognition is what the infinite-valued system G_{χ_0} reflects, constructed by the author as a generalisation of two-valued classical logic, characterising the process of cognition within the framework of operations with the extreme truth values of propositions – truth and falsity. Such is the semantic interpretation of the infinite-valued system G_{χ_0}, disclosing its role in the process of the cognition of truth.

§ 6. The Laws of the Excluded Middle and Non-Contradiction in Non-Classical Logics (Multi-Valued, Intuitionistic and Constructive)

In Chapter IV, "The Fundamental Laws (Principles) of Correct Thought", we analysed the specifics of the operation of the law of the excluded middle, given "indeterminacy" in cognition, and came to the conclusion that this law is applied when cognition encounters a strict situation: either – or, true – false. In many non-classical logical systems, formulas corresponding to the laws of the excluded middle and non-contradiction are not tautologies. Below we present a table in which the sign "+" shows that, in the logical system indicated, the law of non-contradiction and the law of the excluded middle, i.e., the formulas $\overline{a \wedge \bar{a}}$ and $a \vee \bar{a}$, are tautologies (or deduced formulas), and, correspondingly, the sign "–" indicates that they are not. Moreover, we consider the negation of the law of non-contradiction, expressed by the formula $\overline{\overline{a \wedge \bar{a}}}$, and negation of the law of the excluded middle, expressed by the formula $\overline{a \vee \bar{a}}$. These formulas presume the form of negation accepted in the given logical system.

In intuitionistic and constructive logics, the law of the excluded middle "does not work" for infinite sets. In constructive mathematics, "realisability" is understood as the potential realisability of the constructive process, giving as a result one of the terms of the disjunction, which must be true. Since, however, for infinite sets there is no algorithm for recognition what is true: *a* or *not-a*,

Type of logical system	Law of the excluded middle $a \vee \bar{a}$	Law of non-contradiction $\overline{a \wedge \bar{a}}$	Negation of the law of the excluded middle $\overline{a \vee \bar{a}}$	Negation ot the law of non-contradiction $\overline{\overline{a \wedge \bar{a}}}$	Formal contradiction $a \wedge \bar{a}$
1. Two-valued classical logic	+	+	–	–	–
2. Lukasiewicz's three-valued logic	–	–	–	–	–
3. Heyting's three-valued logic	–	+	–	–	–
4. Reichenbach's three-valued logic:					
a) cyclical negation	–	–	–	–	–
b) diametrical negation	–	–	–	–	–
c) complete negation	+	+	–	–	–
5. Post's *m*-valued logic:					
a) first negation	–	–	–	–	–
b) second negation	–	–	–	–	–
6. Markov's constructive logic	–	+	–	–	–
7. Glivenko's constructive logic	–	+	–	–	–
8. Kolmogorov's constructive logic	–	+	–	–	–
9. Heyting's intuitionistic logic	–	+	–	–	–

constructive logic rejects the law of the excluded middle within the framework of constructive mathematics.

Thus, the table shows that the formula $a \vee \bar{a}$, corresponding to the law of the excluded middle, is not a tautology or a proved formula in 10 of the 12 types of negation considered.

The specifics of the law of non-contradiction in non-classical logics

A study of nine formalised logical system disclosed that, out of the 12 types of negation presented, for seven of them the law of non-contradiction is a tautology (or proved formula), while for the other five it is not. Compared with the law of the excluded middle, the law of non-contradiction is firmer.

The law of non-contradiction is not a tautology in many multi-valued logics. In classical, intuitionistic and constructive logics the law of non-contradiction is, on the contrary, recognised as having unlimited operation. The reason is that, in many-valued logics, the number of values of truth may be either finite (more than 2) or infinite. In logical systems where a strict "either-or" (true-false) situation is reflected, the law of non-contradiction and the law of the excluded middle are tautologies. But these are extreme situations in cognition (true or false). If, however, in the process of cognition, we have not yet attained the truth or have not yet refuted some assertive statement (proved it to be false), we have to operate not with true or false statements, but with indeterminate ones.

Classical two-valued logic must be supplemented by multi-valued logics, particularly an infinite-valued logic applicable to reasoning about objects reflected in concepts without a fixed extension, whose infinite number of truth values lies in the interval from 1 to 0.

Constructive and intuitionistic logics reflect completely different situations in cognition: the constructive process either is (is accomplished) or is not, but the two cannot take place at the same time in relation to one and the same constructive object or process, so the law of non-contradiction operates unrestrictedly in these logics. In constructive logics, different abstractions are adopted from those employed in multi-valued logics. Only two values of truth are accepted in constructive and intuitionistic logics – true and false, proved (deduced) or not-proved (not deduced), so the law of non-contradiction is a deduced formula.

Irrespective, however, of whether or not the law of non-contradiction is a tautology in a given logical system, the logical systems themselves are built in a non-contradictory manner: in other words, the theory itself of the construction of formalised systems is subordinate to the law of non-contradiction, otherwise such systems would be pointless, since anything at all, true or false, could be deduced within them.

One very important result on the gnoseological and logical plane is that the law of non-contradiction and the law of the excluded middle cannot be refuted, since the negation of these laws in any known form, in any of the eighteen logical systems studied by the author is not a tautology (or a deduced, proved formula). This testifies to these laws' fundamental role in cognition. The law of non-contradiction is one of the basic laws of correct human thinking. It is firm, and cannot be refuted or replaced by another law, otherwise the difference would be erased in cognition between truth, as its goal, and falsity.

The diversity of logical systems testifies to the development of the science of logic as a whole and of its component parts, including the theory of the basic fundamental formal-logical laws – the law of non-contradiction and the law of the excluded middle.

§ 7. Modal Logics

Classical two-valued logic considered simple and complex assertoric statements, i. e., ones in which the character of the link between subject and predicate is not modified; for example, "sea water is salty" or "rain now began to fall in large warm drops, now stopped completely".

Modal judgements reveal the character of the link between subject and predicate, or between individual simple propositions in a complex modal judgement. For example, "Metals are necessarily conductors of electric current" or "If a fair wind blows, then we may possibly reach harbour before dark".

Judgements are modal when they include modal operators (modal concepts), i. e., the words "necessarily", "possibly", "impossibly", "by chance", "forbidden", "well" and many others (see the section "Division of Judgements According to Modality" in this book).

Modal judgements are considered in a special field of modern formal logic–modal logic.

The study of modal judgements has a long and diverse history, but we shall note only a few of its aspects. Modalities were introduced into logic by Aristotle. The term "possibility" has a variety of meanings, according to Aristotle. He uses it to mean things that are necessary, not necessary or possible. Proceeding from this understanding of the modality "possibility", Aristotle wrote of the inapplicability of the law of the excluded middle to future single events.

Alongside the categorical syllogism, Aristotle also studied the modal syllogism, in which one or both premises and the conclusion are modal judgements. J. Lukasiewicz dedicated two chapters in his book *Aristotle's Syllogistic from the Standpoint of Modern Formal Logic* to Aristotle's modal logic of sentences (Ch. VI) and his modal syllogistic (Ch. VIII).[1] Aristotle regards the modal syllogistic according to the model of his own assertive syllogistic: the syllogisms are subdivided into figures and modi and incorrect modi are discarded with the help of their interpretation in concrete terms. Lukasiewicz gives examples of modi with two apodictic premises (i. e., including the modal operator "necessarily"), which Aristotle provided.

1. If *A* is necessarily not inherent in any *B*, then *B* is necessarily not inherent in any *A*.

2. If, however, *A* is necessarily inherent in all or some *B*, then *B* is also necessarily inherent in some *A*. Aristotle believed that syllogisms with two apodictic premises were identical to the categorical syllogism, with the exception of the fact that the sign of necessity must be added to both premises and the conclusion.

Aristotle also considered modi with one apodictic and one assertive premise, as well as modi with possible premises (problematic statements). He asserts that problematic statements "Every *b* can be *a*" and "Some *b* can be *a*" are both reversible in the sentence "Some *a* can be *b*". (This is a direct inference, called conversion). Lukasiewicz notes that "the laws of conversion of

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[1] Jan Lukasiewicz, *Aristotle's Syllogistic from the Standpoint of Modern Formal Logic*, Clarendon Press, Oxford, 1957.

possible propositions are somewhat negligently treated by Aristotle. He apparently does not attach any great importance to the concept of possibility".[1] Aristotle also studies the laws of conversion of modal judgements, including the modal operator "by chance". According to him, chance is what is neither necessary nor impossible, i. e., *p* by chance means the same as *p* is not necessary and *p* is not impossible, but Lukasiewicz notes that Aristotle's theory of accidental syllogisms is full of serious errors.[2] Lukasiewicz's conclusion is that Aristotle's propositional modal logic is of tremendous significance for philosophy; all the elements necessary for constructing a complete system of modal logic may be found in Aristotle's works; Aristotle proceeded, however, from a two-valued logic,[3] while modal logic cannot be two-valued. Aristotle came very close to the idea of a multi-valued logic when debating the "future sea fight". Following on from Aristotle, in 1920 Lukasiewicz built the first multi-valued (three-valued) logic (we have analysed this above). Such is the interconnection between modal and multi-valued logics.

The philosophers of Ancient Greece, especially Diodorus Cronus, focused considerable attention on elaborating modal categories. Diodorus considered modality in connection with the temporary variable he himself introduced. In the Middle Ages, considerable attention was again paid to modal categories. In the 19th century, the category of probability was developed by Boole and Poretsky.

The emergence of modal logic as a system dates from 1918, when the American logician and philosopher Clarence Irving Lewis (1883–1964) formulated a modal calculus, which he subsequently called S3, in his work *A Survey of Symbolic Logic.*

In the book *Symbolic Logic*, which he co-authored with C. Langford in 1932, he formulated a further five modal logical systems connected with S3 and with one another. These were the systems S1, S2, S4, S5 and S6.

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[1] Lukasiewicz, *Ibid.*, p. 192.

[2] *Ibid.*, Ch. VIII, § 60.

[3] Note that this, now generally accepted term "two-valued logic" was first introduced by Lukasiewicz.

Let us describe modal system Sl[1].

I. Initial symbols. 1) p, q, r, and so on – propositional variables; 2) $\sim p$ – negation of p; 3) $p \cdot q$ – conjunction of p and q; 4) $p \prec q$ – strict implication of the Lewis system; 5) $\Diamond p$ – the modal operator of possibility (p is possible); 6) $p = q$ – strong equivalence, $p = q$ is equivalent to $(p \prec q) \cdot (q \prec p)$.

II. The axioms of system Sl.

1) $p \cdot q \prec q \cdot p$
2) $p \cdot q \prec p$
3) $p \prec p \cdot p$
4) $(p \cdot q) \cdot r \prec p \cdot (q \cdot r)$
5) $p \prec \sim \sim p$
6) $(p \prec q) \cdot (q \prec r) \prec (p \prec r)$
7) $p \cdot (p \prec q) \prec q$

Axiom 5 can be deduced from the others, as was shown later. Since conjunction ties more "strongly" than implication, the brackets may be left out or replaced with dots, as Lewis did.

III. The rules of inference of Sl.

1) The rule of substitution. Any two expressions equivalent to each other are mutually substitutable.

2) Any correctly constructed formula can be substituted for p or q or r and so on in any expression.

3) If p is deducible and q is deducible, then $p \cdot q$ is deducible.

4) If p is deducible and $p \prec q$ is deducible, then q is deducible.

Lewis built the modal propositional logic Sl in the form of an extension of a non-modal (assertoric) propositional calculus (APC for short). Moreover, the main features of Sl and his other calculi were copied from the formal logical system of Russell and Whitehead's *Principia Mathematica*, formulated with the help of concepts that differ only terminologically from those used in the *Principia*. Apart from Russell and Whitehead, the ideas of classical logic have been developed by many modern mathematical logicians, such as the American logician and mathematician S. Kleene[2]. Lewis' calculus is built

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[1] C. J. Lewis and C. H. Langford, *Symbolic Logic*, New York, Dover, 1932, pp. 123–126. In this work brackets are replaced by the sign "·", but we shall use brackets.

[2] S.C. Kleene, *Mathematical Logic*, New York, Wiley, 1967.

axiomatically according to the *Principia* model and, by analogy with the *Principia*, Lewis proves a number of specific theorems.

In classical two-valued logic, logical sequence is identified with material implication and such forms of inference are permitted as: 1) $p \rightarrow (q \rightarrow p)$, i.e., a true statement follows from any statement ("truth follows from anything") and 2) $p \rightarrow (\bar{p} \rightarrow q)$, i.e., any statement follows from a false statement ("anything follows from falsehood"). This contradicts our comprehensive, practical understanding of logical sequence, so the given formulas, as well as certain others and the principles of logical sequence corresponding to them, are called paradoxes of material implication.

Lewis created his own new systems to get round these paradoxes and introduce a new implication, called "strict implication", so that logical sequence might be presented not merely formally, but according to its sense, and the new implication would be closer to the natural language conjunction "if, then". In Lewis's strict implication on $p \prec q$, the antecedent, i.e., p cannot be asserted nor the consequent i.e., q*, negated.

In Lewis's systems, the paradoxes of material implication were eliminated, i.e., the formulas 1 and 2 became non-deducible, but paradoxes of strict implication appeared. These included, for example, such formulas as: 3) $(\sim \Diamond \sim p) \prec (q \prec p)$, 4) $(\sim \Diamond p) \prec (p \prec q)$. Thus, Lewis's strict implication cannot be identified with consequence.

With the aim of excluding Lewis's paradoxes of strict implication, the German mathematician and logician W. Ackermann (1896-1962) built his own system of modal logic. He introduced so-called strong implication, which was not identical to Lewis's strict implication; neither were Lewis's and Ackermann's modal operators the same. Ackermann defines all logical terms and modal operators through strong implication thus: NA is equivalent to $\bar{A} \rightarrow \lambda$, MA is equivalent to $\overline{A \rightarrow \lambda}$. Here A is any correctly constructed formula of the Ackermann system; N is the operator of necessity; M is the operator

* The antecedent is the first term of the implication, preceded by the connective "if". The consequent is the second term of the implication.

of possibility; $\bar{A}$ is negation of A; the sign "$\rightarrow$" designates strong implication. The sign "λ" is the logical constant designating "absurdity". This constant is, in turn, defined thus: $A \& \bar{A} \rightarrow \lambda$, where & designates conjunction. The final formula reads thus: from a contradiction, i. e., A and not-A, follows absurdity.

Formulas structurally similar to the paradoxes of either material implication or strict implication are not deduced in Ackermann's system.

Lewis's and Ackermann's systems are infinite-valued. In contrast to these, the systems initially built by Lukasiewicz are finite-valued: one is three-valued (1920, it is set out on pages 242-246) and the other is four-valued (1953).

Paradoxes have also been discovered in Lukasiewicz's four-valued system.[1]

The chief paradox consists in the fact that not one of the apodictic propositions is true, i. e., no proposition of the form $L\alpha$ (where L designates necessity, and α is any formula) is true. This would mean that there are no necessary propositions, i. e., the modal operator "necessarily" is abolished. Lukasiewicz writes: "As the proposition α may be given any interpretation, $L\alpha$ represents a general law and means that any expression beginning with L, i. e., any apodeictic proposition, should be rejected"[2] Lukasiewicz sees this as an advantage of his system, and regards the concept of "necessity" as a pseudo-concept. We cannot, of course, agree with him on this.

Lukasiewicz posed the problem of extending the four-valued logic into higher systems, and gave the construction of an eight-valued modal logic as an example. The values of truth are 1, 2, 3, 4, 5, 6, 7, 0. The figure 1 designates truth and 0 – falsehood, while Lukasiewicz makes the other figures constitute intermediate values between truth and falsehood. He maintains that, in an eight-valued modal logic, possibilities of different degrees exist, including stronger and weaker ones.[3] Lukasiewicz formulated some problems connected with the further study of modal logics. He writes: "I have

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[1] See: Jan Lukasiewicz, *Op. cit.*, Ch. VII.
[2] *Ibid.*, Ch. VII, p. 170.
[3] *Ibid.*, Ch. VII, p. 80.

always thought that only two modal systems are of possible philosophic and scientific importance: the simplest modal system, in which possibility is regarded as having no degrees at all, that is our four-valued modal system, and the χ_0-valued system in which there exist infinitely many degrees of possibility. It would be interesting to investigate this problem further, as we may find here a link between modal logic and the theory of probability."[1]

Interpretations of modal logics differ. The well-known Austrian philosopher and logician Rudolf Carnap (1891-1970) tried to interpret modal concepts (operators) with the help of the so-called theory of "possible worlds", in which the existence of a set of "worlds" is assumed, one of these being the actual, real world, while the others are possible worlds. That which exists in all the worlds is necessary, while that which exists in at least one of them is possible. That which exists in one of the worlds actually exists. It is usually considered that the term "possible world" was borrowed from Leibnitz, who presumed that the existing world was not the only possible one, but the best of all the possible worlds. Some logicians outside the Soviet Union[2] assert, however, that since the mediaeval Scottish philosopher John Duns Scotus (13th c) made obvious use of the concept of the logically possible state of affairs, in opposition to the real one, he, not Leibnitz, should be considered the author of the idea of possible worlds, this being similar to the modern understanding of modalities in the so-called semantics of possible worlds.

In 1946, Carnap used the concept of a "description of a state" to propose an interpretation of modal operators based on the idea of a difference between the possible and the real world.

In 1959, the American logician Saul Kripke published his first work on the semantics of possible worlds.[3]

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[1] *Ibid.*, Ch. VII, p. 180.

[2] See: S. Knuuttila, "Time and Modality in Scholasticism".–In: *Reforging the Great Chain of Being: Studies of the History of Modal Theories*, Dordrecht (Holland), D. Reidel Publishing Comp., 1981, pp. 163–257.

[3] Kripke S., "A Completeness Theorem in Modal Logic". In: *The Journal of Symbolic Logic*, Vol. 24, No 1, 1959, pp. 1–14.

Relational semantics of possible worlds are often called Kripke-type semantics. Similar results were, however, obtained by a numer of other investigators independently of and parallel to him. The works of B. Jónsson and A. Tarski[1] are among the algebraic developments of the idea of the semantics of possible worlds.

The Finnish logician Jaakko Hintikka proceeded in another direction. After critically rethinking the concept of "description of a state" introduced by Carnap, he developed a technique of "modal sets", i. e., worlds (1957), an original semantic conception of possible worlds. The elaboration of the semantics of possible worlds for modal logics is still in process.

The American logician Robert Feys is engaged in diverse problems of modal logic.[2]

Saul Kripke, in his work *Semantical Analysis of Modal Logic I. Normal Modal Propositional Calculi*[3], formulates a modal calculos of statements (MCS), which is given by an infinite list of propositional variables *P*, *Q*, *R*, . . . , which can be combined using the connectives $\wedge$, $\sim$, $\Box$ (designating conjunction, negation and necessity respectively) to give correctly constructed formulas. Propositional variables are thus atomary formulas of the systems described. Later, Kripke uses the letters *P*, *Q*, *R*, . . . as metavariables, taking the values of the set of atomary formulas, and the letters *A*, *B*, *C*, . . . as the metavariables taking the values of the set of arbitrary formulas. Kripke calls the modal calculus *normal* if it contains, as theorems, schemes of the axioms A1 and A3, and if the two rules R1 and R2 are taken as the permissible (deducible) rules of inference.

A1. $\Box\ A \supset A.$
A3. $\Box\ (A \supset B) \supset (\Box\ A \supset \Box\ B)$
R1. If $\vdash A$ and $\vdash A \supset B$, then $\vdash B$.
R2. If $\vdash A$, then $\vdash \Box\ A$.

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[1] Jonsson B., Tarski A., "Boolean Algebras with Operators".– In: *American Journal of Mathematics*, Vol. 73, Baltimore, Hopkins Press, 1951, pp. 891–939.

[2] R. Feys, *Modal Logics*, Louvain. Nauwelaerts, Paris, Gauthier-Villars, 1965.

[3] S. A. Kripke, *Semantical Analysis of Modal Logic I, Normal Modal Propositional Calculi*, ZMLGM, 9, 1963, pp. 67–96.

(In non-normal Kripke's systems, rule R2 is not fulfilled; here $\supset$ is implication and $\vdash$ the sign of conclusion.)

The system M (or T) of Feys (1937-1938) and Von Wright G. H. (1951) is set by the axioms A1 and A3 and the rules R1 and R2. Lewis's system S4 is obtained from M by adding the schema of the axiom A4. $\Box A \supset \Box \Box A$.

Brouwer's *axiom* is the schema $A \supset \Box \Diamond A$.

Brouwer's *system* is obtained from M by adding Brouwer's axiom.

Finally, Lewis's system S5 is defined as M plus the schema A2. $\sim \Box A \supset \Box \sim \Box A$.

It is known, Kripke writes, that S4 plus Brouwer's axiom is equivalent to S5.[1]

Thus, we can see that, in the work written in 1963, Kripke showed the interconnection between a number of modal propositional calculi, formulated previously by the logicians Lewis, Feys and Von Wright G. H., and also Brouwer.

Many types of modalities have so far been elaborated that are reflected in the schema on page 86 of this textbook.

The Soviet logicians A. A. Ivin,[2] Ya. A. Slinin,[3] O. F. Serebryanikov, V. T. Pavlov and others are actively engaged in studying the theories of modal logics and the construction of new modal logical systems.

§ 8. Positive Logics (Negationless Logics)

Logics are called positive if they are built without the operation of negation. Such a definition of positive logics is given, for example, in the works of the American logicians Alonzo Church and Haskell Curry,[4]

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[1] *Ibid.*, p. 255.

[2] Ivin A. A., *Foundations of the Logic of Assessments*, Moscow, 1970; *The Logic of Norms*, Moscow, 1973 (in Russian).

[3] Slinin Ya. A., *Modern Modal Logic*, Leningrad, 1976 (in Russian).

[4] Alonzo Church, *Introduction to Mathematical Logic*, Vol. 1, Princeton, New Jersey, Princeton University Press. 1956; Haskell B. Curry (Evan Pugh Research Professor in Mathematics, The Pennsylvania State University), *Foundations of Mathematical Logic*, McGraw-Hill Book Company, New York,1963.

the Poles Helena Rasiowa and Roman Sikorski[1], and the Germans David Hilbert and Paul Bernays.[2]

The author considers it expedient to divide the set of all positive logics into two types, because these types have a number of distinguishing properties.

The first type. Positive logics in the broad sense of the word, or quasi-positive logics. They are built without the negation operation, but negation can be expressed within them by the means of this logical system.

The second type. Positive logics in the narrow sense of the word, built without the negation operation, and in which negation cannot be expressed.

Common to both types of positive logic is the fact that the logical constants of these systems do not include a negation operation.

Another classification of positive logics may be offered that is based on the number of operations on which the positive logic is constructed: one, two, three or more.

First we shall analyse quasi-positive logics.

The quasi-positive logics based on a single operation include the following: the logic based on "Sheffer's stroke"* (anticonjunction) and the corresponding binary logic based on the operation of antidisjunction. Analogies of "Sheffer's stroke" also exist in constructive logic.

A number of quasi-positive logics are based on two operations.

Positive logics in the narrow sense

The positive logics based on a single operation include the following: 1) implicative logic, based on the operation of implication; 2) logic constructed on the operation of equivalence.

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[1] Helena Rasiowa and Roman Sikorski, *The Mathematics of Metamathematics*, Polska Akademia Nauk, Monografie Matematyczne, Vol. 41, Państwowe Wydewnictwo Naukowe, Warsaw, 1963.

[2] David Hilbert, Paul Bernays, *Grundlagen der Mathematik*, Vol. 1, Berlin, Springer, 1934.

* The operator "Sheffer's stroke", introduced by H. Sheffer in 1913, is written in the form *a*/*b* and means "*a* and *b* are incompatible" or "it is untrue that *a* and *b*".

A number of positive logics are based on two operations: a) on implication and conjunction; b) on disjunction and conjunction; c) on implication and disjunction.

The positive logics based on three or more operations without negation include Hilbert's positive propositional calculus (P^p), in which four basic connectives are: implication, conjunction, disjunction and equivalence, the rules of inference – *modus ponens* and substitution; axioms are 11. The basic connectives are not independent and they do not include negation.

H. Curry's NA and LA systems, propositional algebras built on the operations of implication, conjunction and disjunction, are among the positive logics based on three operations. In his book *Foundations of Mathematical Logic*, Curry includes a section entitled "Classical Positive Propositional Algebra", in which he considers classical positive propositional systems NS and TS, formed with the help of implication, conjunction and disjunction.

Let us analyse the correlation between positive (in the narrow sense), intuitionistic and classical logics.

A positive logic (in the narrow sense) is a subsystem of stronger logics – intuitionistic and classical, or, in other words, positive logic is a partial system. All the provisions of positive logic also apply in intuitionistic logic and in classical logic. Within positive logics there are also systems of various strength: thus, for example, implicative logic, including two axioms, is weaker than positive logic including, in addition to these two, further axioms characterising conjunction and disjunction.

The axiomatic construction confirms this correlation: the strongest is the classical; intuitionistic is weaker; and positive logic weaker still; the first is built on 12 axioms, the second on 11 and the third on nine (of the same twelve).[1]

The distinction between positive logics in the broad sense (quasi-positive logics) and positive logics in the narrow sense consists in the following:

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[1] See axioms in the book: H. Rasiowa and R. Sikorski, *The Mathematics of Metamathematics*, Warsaw, 1963, p. 169.

1) In quasi-positive logics the operation of negation can be expressed by the means of this logic, while the operation of negation cannot be expressed in positive logics in the narrow sense.

2) Quasi-positive logics are models of classical logic, i. e., they are equivalent to the classical propositional logic. Positive logics are not equivalent to classical logic; they are a subsystem (partial system) of it, so, consequently, are weaker than the classical propositional logic.

The role of positive logics in artificial languages

The constructive logic of the Soviet mathematician A. A. Markov (1903-1979) is built on hierarchy of languages. The alphabet of language L_1 contains no negation and it cannot express negation, for there is no implication. Although language L_1 is narrow, a theory of normal algorithms may be constructed with its help. Markov writes that he built "language L_1, suited to describing the work of normal algorithms".[1] This language is suited to expressing some relations between words encountered in pure semiotics and in the theory of normal algorithms. Markov formulated the principle of normalisation: "any algorithm can be normalised". Consequently, the following conclusion may be made: if, with the help of language L_1, a theory of normal algorithms may be constructed, and any algorithm can be normalised, various algorithms can be constructed with the help of language L_1 (a language without negation), and this constitutes the major significance of a language without the operation of negation.

Furthermore, the method of non-formalised programming for digital computers, which has become widespread because it combines a number of the advantages of programming in machine language and programming in algorithmic languages, contains, among the logical operations that constitute a special group of commands

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[1] Markov A.A., "On the Language L_2", *DAN USSR*, 1974, V. 214, No. 3, p. 513.

to the machine MM, only logical addition and logical multiplication; the operation of logical negation is absent.

Consequently, a logical language without the operation of logical negation can be used in building machine programmes.

If we take such artificial languages as FORTRAN, COBOL and others that allow a highly efficient method of programming to be used, these languages contain, apart from logical addition and logical multiplication, also logical negation, corresponding to the particle "not" and designated by the usual sign "7".

All instructions on how to assemble machines, instruments, technical instruments, furniture and so on are based on the non-formalised rather than formalised use of positive logics.

In a number of his works, the Dutch logician G. Griss[1], beginning from 1944, proposed building an intuitionistic mathematics without negation. Griss believed that only negationless construction has some sense in intuitionistic mathematics, so he rejects the use of negation as a mathematical concept. Griss only indicated general principles and gave a number of examples, but he offered no developed system for building a negationless intuitionistic mathematics. The intuitionist A. Heyting believes that Griss attained some remarkable results in his attempt to rebuild intuitionistic mathematics without using negation. In contrast, Thoralf Skolem proved that no intuitionistic theory of sets can be built in a negationless system. Paul Gilmore subjected Griss's attempt to profound criticism.

How justified are attempts to build negationless mathematics and logic? In order to answer this question, let us consider how Griss made his attempt. Rejecting negation, he replaced it with the relation of distinction, which he designated by the sign "$\neq$". Griss builds a series of natural numbers, using the relations of "distinc-

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[1] G. F. C. Griss, "Negationless Intuitionistic Mathematics", *Koninklijke Nederlandse Akademie van Wetenschappen, Proceedings*, Vol. XLIX, No. 10, 1946; Vol. LIII, No. 4, 1950; Vol. LIV, 1951; G. F. C. Griss, "La mathematique intuitioniste sans négation", *Nieuw Archif voor wiskundo* (3) III, 1955, p. 134.

tion" and "identity". The intuitionist Brouwer considers the predicate "$\neq$" to be, in essence, negative. Moreover, Griss introduced the concept of the complement of a class[1] into the logic of negationless intuitionistic mathematics.

Griss did not succeed in his designs, since he introduced into the "negationless logic of classes" the complement of class *A*, i. e., the analogue of negation, since the complement to class *A*, designated by *A'*, is the negation of class *A* (as shown in the topic "The Concept" in the section headed "Operations with Classes").

The French logician Paulette Destouches-Février also made an attempt, in 1947, to build a positive logic. Fevrier proposed her own conception, supposedly intermediate between Brouwer's intuitionism and Griss's negationless intuitionism. She suggests that logic should be regarded as a component part of mathematics. In her "logic of positive intuitionism", she adopts a certain concept of negation, provided it is defined completely by positive methods. She proceeds from Griss's theory of natural numbers in which, in her opinion, two binary relations between natural numbers are defined in a positive way: "$=$" (identical) and "$\neq$" (distinct). The concept of contradiction must also be defined without negation, by positive means. Destouches-Février calls the proposition $\emptyset$ defined in the following way a contradiction:

$$\emptyset \underset{Df}{=} (a = b) \text{ and } (\mathrm{a} \neq \mathrm{b}) \text{ or } \emptyset = (1 = 2),$$

where *a* and *b* are whole numbers.

Correspondingly, a theory will be contradictory if the truth of $\emptyset$ can be established within it.

Février proposes the name intuitionistic positive mathematics for her conception, intermediary between Griss's negationless mathematics and Brouwer's and Heyting's intuitionistic mathematics, and quite close to the mathematics outlined by the Norwegian mathema-

[1] G. F. C. Griss, "Logic of negationless intuitionistic mathematics", *Koninklijke Nederlandse Akademie van Wetenschappen, Proceedings,* Vol. LIV, No. 1, 1951, p. 41.

tician Ingmar Johansson[1]. In her opinion, Février defines negation by "positive means", proceeding from the concept of contradiction: $7\,p \underset{Df}{=} p \rightarrow \emptyset$, where $\emptyset$ is the proposition meaning contradiction and defined positively in each deductive theory, for example, $\emptyset = (1 = 2)$.

Negation is defined in the same way in Heyting's intuitionistic logic. The Soviet mathematician A. N. Kolmogorov interpreted negation analogously: "7 *a*" means the problem: "Assuming that the solution *a* is given, find its contradiction". One of the negations in the constructive logic of A. A. Markov is defined similarly, as is Glivenko's negation, as well as negation in classical logic. Yet none of these logical systems are considered to be positive calculi (logics) so we consider Février's assertion that she has constructed a "positive" logic rather dubious.

§ 9. Paraconsistent Logics

One of the forerunners of paraconsistent logics was the Russian logician N. A. Vasilyev. Back in 1910-1913 he formulated the idea of the possibility of an axiomatic logic and of the non-uniqueness of logic. Realising the latter idea, in 1912 he built an "imaginary" non-Aristotelian logic with the law of the excluded fourth and without the law of contradiction, which he formulated thus: "No thing has a predicate contradicting it".[2] An "imaginary" logic describing an imaginary world will contain a different negation from Aristotle's. In the new logic, Vasilyev adopts three types of propositions: affirmative of the type "*S* are *P*", negative of the type "*S* is not *P*" and, in Vasilyev's terminology, indifferent propositions of the type "*A* are and are not *P*". The law of the excluded fourth is necessary in this logic, prohibiting

[1] Paulette Destouches-Fevrier, "Ésquise d'une mathématique intuitioniste positive", *Comptes rendus. Hebdomadaires des séances de l'académie des sciences,* 1947, Vol. 25.

[2] N. A. Vasilyev, "Logic and Metalogic", *Logos,* Kazan, No. 1–2, 1912–1913, p. 62 (in Russian).

the formation of a fourth type of proposition (analogously to the law of the excluded middle of the two-valued logic).

Vasilyev suggests that, in Aristotle's logic, the negative statement and the indifferent statement of the "imaginary" logic merge into a single, negative one. In Vasilyev's logic, the former (Aristotelian) negation is called "absolute", while the latter negation is called "relative" ("negation in the restricted sense"). The American logician G. Kline considers Vasilyev's distinction between negations to be similar to Post's distinction between complete and incomplete falsehood, while he regards Vasilyev's logical ideas to be connected with the ideas of multi-valued logics. Kline's opinion on this is quite convincing.

Even Aristotle pointed out in his *Posterior Analytics*, that the law of non-contradiction or consistency (some logicians call it the "law of contradiction") is not universally applicable when he spoke of the independence of the principle of the syllogism from the law of consistency. Concerning future single chance events, Aristotle believed that a proposition could not be stated to be true or false; that it was indeterminate. Consequently, if *A* is a proposition concerning a future chance event, then *A* and not-*A* can both be true, i.e., a temporal paraconsistent semantics is presumed. From the law of consistency it follows that two opposing propositions cannot be true at one and the same time. At different times, however, the situation changes. In the course of a single period of time, the statement *A* is true, while during another its negation, i.e., not-*A* is true. In accordance with this understanding of contradiction, Aristotle analyses propositions about changes: "everything that changes must be divisible ... part of that which is changing must be at the starting point and part at the goal: for as a whole it cannot be in both or in neither".[1]

As a consequence of the fact that transitional states exist, however, it is not possible to draw a precise

[1] Aristotle, *The Works of Aristotle*, Chicago, William Benton, Publisher; Encyclopaedia Britannica, Inc., 1952, Vol. I, *Physics*, Book VI, Chapter 4, p. 316.

dividing line between the states before and after many types of change. Indeterminacy appears in the intervals and states of that which is changing, so, during the period of the transitional state, both statement *A* and its negation, statement not-*A*, are true.

Analysis of imprecise, "fuzzy" sets, with no fixed extension, shows that, for these sets, neither the law of consistency nor the law of the excluded middle work. In scientific and everyday thinking, people often have to analyse concepts that are flexible, dialectical and mobile. In connection with analysis of these properties, the problem has arisen of formalising the "non-rigid" concepts applied in the humanities.

In philosophical literature, the problem of formalisation is usually studied in application to mathematics, symbolic logic, cybernetics and other sciences, which make use of concepts with a "rigid", fixed extension, and apply algorithms that precisely prescribe the sequence of operations with these concepts. In the process of reflecting objective reality, however, we also have to operate in our thoughts with concepts that are dialectically flexible, come across so-called "fuzzy" algorithms, deal with methods for solving problems that are not precisely formulated, and operate with flexibly formulated conditions. The study of the specific aspects of operating with such "non-rigid" thought objects will also promote further research connected with the transfer of certain intellectual functions to computers.

Analysis of concepts in "fuzzy" sets engendered the need to supplement the classical two-valued logic applied for concepts with "rigid" extensions and for statements with the values of truth 1 (true) and 0 (false), with an infinite-valued logic applied in reasoning on "non-rigid" objects and reflecting their concepts with a non-fixed extension (an infinite value of truth lying in the interval from 1 to 0). The new theory of "non-rigid" objects is applied in intellectual-cognitive and intellectual-practical activities, namely in the sphere of decision-making, in certain aspects of the modelling of mental processes, i. e., in building a theory of "fuzzy" algorithms, a theory of "fuzzy" graphs, the introduction of the concept of the "fuzzy" automatic machine, and others.

The conception of non-rigid (fuzzy) sets was suggested in 1965 by the American mathematician L. A. Zadeh[1], whose ideas are shared by the well-known modern mathematician Richard Bellman.

Thus, the possibility of paraconsistent logics emerging was engendered by the requirements of scientific analysis of transitional (temporal) states, non-rigid sets and a number of other analogous logical problems.

Logical calculi that can provide the basis for contradictory formal theory have become known as "paraconsistent" logics. The sphere of application of logics in which contradiction is localised, i. e., not everything can be deduced by the means of such a logic, is very broad. They are used for building a paraconsistent theory of sets and studying paradoxes. In discussions, scientific polemics and court cases contradictory points of view arise, but this does now usually prevent the correct solution being found to the issue under discussion or to the case being heard. Contradictory empirical data sometimes emerge, but this does not mean that the theory is discarded in its entirety. The problem arises of constructing information systems that could operate with contradictory data.

The system of paraconsistent logic must, in the general case, satisfy the following conditions:

1) From the two contradictory formulas A and $7A$, in the general case, the arbitrary formula B cannot be derived.

2) The deductive means of classical logic must be retained to the maximum, since they constitute the basis of all common arguments.

In the first place *modus ponens* must be retained, i. e., reasoning according to the formula $((a \rightarrow b) \land a) \rightarrow b$.

The following three concepts should also be introduced: "contradictory theory", "trivial theory" and "non-trivial theory".

Let T be a deductive system, i. e., a theory based on some logic. T will be a trivial theory (or, in A. Church's terminology an "absolutely contradictory" one) if all the

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[1] L.A. Zadeh, "Fuzzy Sets".– In: *Information and Control*, Vol. 8, No. 3, 1965.

formulas of T are theorems of T; otherwise, we say that T is a non-trivial theory. T is a contradictory theory if it contains a symbol of negation and at least two theorems such that one of them is a negation of the other, otherwise it is called consistent.

The problem consists in constructing logics that might provide the basis for contradictory, but non-trivial theories. Classical, intuitionistic, constructive and a number of other logics are no good for this purpose. Positive logics are no good either, since they contain no negation operation.

Thus, a propositional logic must be built that, when applied to a contradictory theory, does not engender triviality of the latter.

Systems of paraconsistent logics are being studied by the Polish logician Stanislaw Jaśkowski, the Brazilian logicians A. I. Arruda[1] and N. C. A. da Costa, and a number of others. The Soviet philosophers A. S. Karpenko, V. A. Smirnov, V. M. Popov, A. T. Ishmuratov and others are currently engaged in building paraconsistent logics.

It has been established that a paraconsistent logic is connected with many types of non-classical logics: with modal logic (i.e., Lewis's S5 system), with multi-valued logics, and with relevant logic, where the principle that "anything follows from a contradiction" is also not applied.

A. S. Karpenko has suggested building a paraconsistent multi-valued logic and has formulated its properties.

The author of the present textbook has shown that the law of consistency, i.e., the formula $\overline{a \wedge \bar{a}}$, is not a tautology or a deduced formula in many logical (namely multi-valued) systems, i.e., these logics get by without the law of consistency. These are: Lukasiewicz's three-valued logic, Reichenbach's three-valued logic (for cyclical and diametrical negations), Post's m-valued logic, Goodstein's three-valued logic, and Bochvar's three-valued logic (for internal negation). In general, the study of 13 formalised logical systems has revealed that,

[1] A. I. Arruda, "A Survey of Paraconsistent Logic".– In: *Mathematical Logic in Latin America*, 1979, pp. 1–41.

out of the 17 types of negation presented, for ten of them the law of consistency is a tautology (proved formula), while for the remaining seven it is not. This is partially reflected in the table on page 318), but this does not include Goodstein's nor Bochvar's three-valued logics, investigated by the author.

This occurs in multi-valued logics because they contain the "indeterminate" value of truth. In classical, intuitionistic and constructive logics, a rigid "either – or" situation is reflected (true-false), so that law of consistency is a tautology (proved formula), i. e., it cannot be rejected.

Let us stress once more that in none of the 18 logical systems studied by the author is the formula $a \wedge \bar{a}$ a tautology (proved), i. e., a formal contradiction, nor is the formula $\overline{\overline{a \wedge \bar{a}}}$ i. e., negation of the law of consistency, a tautology. Thus, in these systems, the law of consistency itself and the negation of the law of consistency are not identically true (or proved) formulas. Metalogic and metatheory are constructed with the use of the law of consistency.

Conclusion

If a person is asked: "Is the sun beneficial or harmful?" the answer will be that it is, of course, beneficial, that it is essential to mankind and without it there would be no life on Earth. Yet too much sun can cause sunstroke, and in agriculture droughts destroy crops. So everything depends on its measure, i. e., on the conditions under which the sun is beneficial and under which it is harmful. The answer itself requires a dialectical approach and dialectical thinking. Moreover, the question was not formulated correctly or dialectically.

In his work "On the Question of Dialectics", Lenin considered the opposition between dialectics and metaphysics. Dialectics and metaphysics are two opposing approaches that arose in ancient times in philosophy to considering nature, society and cognition. Dialectics and metaphysics are two different concepts and understandings of development. Dialectics regards development as a unity of opposites (i. e., divarication of the whole into mutually exclusive opposites and the interrelationship between them), while metaphysics understands development as reduction and increase, as reiteration. Dialectics focusses most attention on cognition of the source of development – self-movement. Metaphysics does not regard self-movement as the source of development, but transfers it outside – it is God, the subject, and so on. Dialectics shows that development takes place in leaps and bounds, through interruptions in gradual advancement, and leads to the elimination of the old and the emergence of the new. The dialectical conception is vital, while the metaphysical one is dead, palid and arid.[1]

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[1] V. I. Lenin, "On the Question of Dialectics", *Collected Works*, Vol. 38, pp. 357–61.

In simple, rather than philosophical terms, to think dialectically is not to invent that which does not exist in reality, but to recognise the development of the material world that exists objectively, i.e., outside us and independently of us. Objective dialectics is itself the "logic of things", i.e., the logic of the development of things themselves, phenomena, and events. A simple example can illustrate this: if milk boils and we do not take it off the heat fast enough, it boils over. If a sick person is not treated, the development of the illness will either lead to disruption of the functions of the organism or to death.

Alongside objective dialectics, there exists subjective dialectics, which is the reflection of the objective dialectics in our consciousness. In a person's mind (the subject), in his thoughts and images, that which exists and operates outside us and independently of us is reflected in a particular way. Ultimately, the two coincide.

Dialectics has gone through many changes. Initially it was the "spontaneous dialectics" of the ancient philosophers; in the 18th century Hegel developed his idealistic dialectics. Marx, Engels and Lenin elaborated a materialist dialectics by combining materialism and dialectics. The classics of Marxism showed convincingly that "An exact representation of the universe, of its evolution, of the development of mankind, and of the reflection of this evolution in the minds of men, can therefore only be obtained by the methods of dialectics with its constant regard to the innumerable actions and reactions of life and death, of progressive and retrogressive changes".[1]

Dialectics shows that the true motive force behind the development of nature, society and cognition are opposites and their interaction – unity and struggle. Subjective dialectics, understood as the image of objective dialectics, reflects the contradictions inherent in the objective world.

In people's thinking, contradictions can be both dialectical (for example, the contradiction between a previous scientific theory and new experimentally ob-

[1] F. Engels, *Anti-Dühring*, p. 31.

tained facts, contradictions between two competing hypotheses or theories, and so on) or formal logical. Dialectical contradictions that arise in the process of cognition are the motive force behind cognition, the inner stimulus to theoretical thinking; formal logical contradictions should be avoided, since they lead to incorrect thinking.

Throughout the history of the development of formal logic, the operation of the three laws of material dialectics have been considered: the law of unity and struggle of opposites, the law of the mutual transformation of qualitative and quantitative changes, and the law of the negation of negation. We shall show in more detail how the law of the unity and struggle of opposites emerged in the history of traditional formal logic and in the theories of modern symbolic logic.

Dialectics proceeds from the fact that there are internal contradictions inherent in the objects and phenomena of nature, society and cognition, that they have a positive and a negative aspect, a past and a future, moribund and developing parts. These opposing aspects, forces and trends in objects and phenomena, negating and, at the same time, mutually conditioning each other, are opposites. Opposites cannot exist without each other: in cognition they are the old theory and the new, the sensual stage of cognition and abstract thinking, and so on.

Characteristic of the initial stage in the establishment of traditional logic was its interconnection with rhetoric, for thinking and speech are inseparable. Throughout the development of logical teachings, we have seen their close interconnection with philosophy as a science, as well as with the philosophical views of individual scholars. The history of logic, even set out briefly, makes it possible to trace the change and development of our knowledge of deductive conclusions (in particular, of the categorical syllogism), of induction, concepts and judgements (including modal ones), the laws of formal logic, the theory of logical sequence, the theory of semantic paradoxes and many other problems of formal logic. We have seen that the means for resolving these problems are not of equal strength in traditional and modern mathematical logic: the symbolic apparatus of the latter allows many logical problems to be reset and solved anew. The law of the negation of negation operates here in that there is often a return, as it were, to old problems, but these are resolved at a higher level and include the positive achievements of the previous periods.

The law of the unity and struggle of opposites is one of the most important laws of materialist dialectics, operating in nature, society and cognition. It also appears in the development of formal logical theories: the unity of such opposites as induction and deduction, analysis and synthesis, as identity and diversity, assertion and negation, the null and universal classes, generalisation and restriction of concepts, truth and error (falsehood), dual value and multiple value of the truth values of statements, the finite nature of the values of truth and their infinity, the identically true formula (i.e., a law of logic) and the identically false formula (formal logical contradiction), proof and refutation, confirmation of hypotheses and refutation of hypotheses, the existence of logics built with the operation of negation and of positive logics and the presence of many other opposites testifies that formal logic is subordinate in its structure, functioning and development to the general law of materialist dialectics – the law of the unity and struggle of opposites.

In Chapter VIII we considered in detail the views of Boole, Jevons, Schröder, and Poretsky on the problem of negation, since the operation of negation is a basic one in logic. The dialectics of the interconnection between negation and other operations of logic, of the interconnection and role of assertive and negative statements in cognition illustrates the specifics of the way the law of the unity and struggle of opposites appeared in formal logic.

The dialectics of assertion and negation was manifested in Boole's works, for example, in his recognition of the unity of addition and subtraction, which he called converse (opposite) operations. He understood addition as the unification of the parts in the whole, and subtraction, on the contrary, as the separation of part from the whole. The dialectics of negation and assertion is manifested, according to Boole, in the fact that assertion cannot be expressed without negation, that a whole series of negations can be derived from a single affirmative statement.

The views of Boole and Schröder, who recognised opposite operations (i. e., subtraction and division) for the operations of addition and multiplication in logic, clashed, however, with those of Jevons and Poretsky, who did not recognise the operations of subtraction and division. The struggle between the opinions of these scholars, and the presence of dialectical contradictions in cognition, as a manifestation of the law of the unity and struggle of opposites, was shown in Chapter VIII. As a result of the analysis of these opposite approaches, a synthesis of knowledge is given – the author's position on the operation of subtraction. This is how scientific views develop from the simple to the complex, from incomplete to more complete knowledge, and so on.

Attempts in logic to separate and disunite opposites, leaving one and discarding the other, have failed. This was demonstrated in describing Griss's attempt to build a so-called intuitionistic mathematics without negation.

A multitude of examples has been given above of interconnected opposites studied in formal logic, including proof and refutation. They are hard even to imagine separately, for in the process of refuting someone else's opinion (which seems false to us), we automatically formulate our own and strive to prove it. This often leads to discussions, arguments and polemics, as a result of which we often obtain true knowledge, although polemics frequently fail to produce any results.

The unity of the opposites of the generalisation and limitation of concepts allows us to study the intension and extension of the initial concept in more depth.

Many of the interconnected and mutually-conditioning opposites listed apply to the construction of various systems. Thus, not only the principle of dual, but also that of multiple truth values of propositions engendered in the 20th century multi-valued (finite-valued) logics of Lukasiewicz, Heyting, Reichenbach, and Post, as well as the three-valued logics of Goodstein, the Soviet logician Bochvar, the American logician Kleene and a number of other multi-valued systems not considered in this book. The interconnection between finite-valued and infinite-valued logical systems is clear from the logics of

Lukasiewicz, who built three-valued, four-valued and infinite-valued logics. From the author's infinite-valued logic G_{χ_0}, any infinite number of finite-valued logical systems can be obtained: three-valued, four-valued, five-valued, ..., twenty-valued, i. e., any *n*-valued logical system. This is only a matter of technical skill and time, for the algorithm is given for building these systems, proceeding from the infinite-valued logic.

In formal logic, the law of the unity and struggle of opposites is also manifested in the presence of a multitude of systems: the two-valued classical (traditional) logic, multi-valued, constructive, modal, including deontic logics (containing the modal operators "essential", "prohibited" "permitted") and temporal logics (including the operators "always", "only sometimes", and "never", as well as the comparative operators: "earlier", "simultaneously" and "later"). The search for new logical systems is still going on (for example, in relevant logics, which closely relate the meaning of the connectives "if, then" to that used in natural language), in paraconsistent logics and others. There can be an infinite set of logical systems for embracing and reflecting the developing, infinite world. Multi-valued logics include an infinite number of finite-valued systems and several infinite-valued ones.

Thus, we have seen the diverse and original ways in which the law of the unity and struggle of opposites operates in the science of logic and in thinking, the laws of which it reflects.

Let us consider another principle of material dialectics – the idea of development – as applied to formal logic. Throughout the history of logic, its basic and secondary ideas have been constantly developing, old theories have been replaced by new, improved ones, methods for investigating logical problems have been changing, the apparatus for formalising and using symbols developing from the simplest to the powerful, ramified apparatuses of modern mathematical (symbolical) logical systems. These have been shown using examples of multi-valued, constructive, intuitionistic, modal and positive logics. This development will, of course, continue and probably gain impetus.

The development of all cognition, logic included, is manifested in the impossibility of fitting the whole logic of human thinking within a single complete system. Leibnitz's attempt to fulfil his impossible dream of replacing all reasoning by calculi was a failure not because of any particular errors made by this great mathematician, but because it is impossible to halt the development of logic by enclosing it once and for all within a set system.

The similar desire of the logicists Frege, Russell and others to reduce all mathematics to logic (and, at the same time, to consider logic as an a priori science) and thus substantiate mathematics, was unsound because account must be taken of the dialectics of mathematics. Mathematics as a science is developing rapidly, becoming enriched by more and more new sections, of both theoretical and practical significance, including for the purposes of computer programming.

At this point I should like to stress once more the methodological inferences drawn from Gödel's theorems of the incompleteness of formalised arithmetic, from which it follows directly that the definition of mathematical concepts in terms of "logics", although this does reveal certain links between these concepts and logic, does not deprive them of their specific mathematical content. A formalised system has sense only in the presence of a non-formalised scientific theory that the given formalised system might serve to systematise.

The non-formalised scientific system (i.e., model) is primary, while its formalisation is secondary. In the gnoseological correlation between a formalised system and its substantive model, the model is primary, not its formalisation. True, once the formalisation has been carried out, an "inversion of the method" takes place, to use Marx's words on a similar matter in his *Mathematical Manuscripts*: the secondary becomes primary, and we begin to seek new models for the axiomatic (or otherwise formalised) system we have obtained from reflecting one of these. This is the case, for example, with the various models of Euclid's geometry. In contrast, in the history of Lobachevsky's geometry, the creation of the deductive scientific theory preceded the finding of models. Quite characteristic of modern mathematics (and logic), in which the axiomatic method plays a major role, is the transfer over recent years of the centre of gravity from research into the internal properties of axiomatic systems to research into their relationships with models, i. e., to what these formal systems reflect or are capable of reflecting, in other words, that to which they may be applied. In solving this question, the "inversion of the method" can be of quite fundamental significance.

Dialectics teaches that there is no abstract truth; that truth is always concrete, this proposition is confirmed repeatedly in formal logic. Alhough logic studies the most general laws of correct thinking, applied in any sphere of science or everyday thinking, these laws have to be studied specifically, i. e., in their dependence on the properties of the object spheres they reflect. Thus, in mathematics it is not always possible to use the law of the excluded middle (as we have shown in detail), nor to consider a concept to have a fixed extension (for example, the concepts "young father" or "interesting textbook" have no such fixed extension), nor to replace negation of a connective with inclusion of the subject in the complement to the predicate[1], nor to consider two equi-extensive concepts as mutually substitutable[2].

We have shown some (though not all, of course,) manifestations of materialist dialectics in formal logic.

Dialectical logic as a teaching on the laws and forms of development of scientific cognition also studies thought processes, but on a different plane from formal logic. Let us consider the definition of dialectical logic and its correlation to formal logic.

Lenin wrote that logic, dialectics and the theory of cognition are "one and the same thing". "In *Capital*, Marx applied to a single science logic, dialectics and the theory of knowledge of materialism (three words are not needed: it is one and the same thing) which has taken

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[1] For example, "I am not happy" does not mean that "I am unhappy".

[2] If, in the statement "The author of the novel *Resurrection* is L. N. Tolstoy", we replace the concept "the author of the novel *Resurrection* by its equivalent concept "L. N. Tolstoy", we obtain the tautology "L. N. Tolstoy is L. N. Tolstoy".

everything valuable in Hegel and developed it further".[1] Lenin evidently meant dialectical logic here. In this sense, dialectical "logic is the science of cognition. It is the theory of knowledge".[2] Lenin formulated the principles of dialectical logic as the methodology of scientific cognition: "Firstly if we are to have a true knowledge of an object we must look at and examine all its facets, its connections and 'mediacies'. That is something we cannot ever hope to achieve completely, but the rule of comprehensiveness is a safeguard against mistakes and rigidity. Secondly, dialectical logic requires that an object should be taken in development, in change, in 'self-movement'... Thirdly, a full 'definition' of an object must include the whole of human experience, both as a criterion of truth and a practical indication of its connection with human wants. Fourthly, dialectical logic holds that 'truth is always concrete, never abstract'".[3]

Dialectical logic and formal logic study thought processes differently; in particular, the former considers the forms of thought as they develop, in their subordination, while the latter, reflecting the presence of relative calm in the world, considers them as already established, abstracting them from movement and change. While recognising the movement and development of objects, of course, formal logic still demands that every thought in the process of reasoning be identical to itself, otherwise logical errors of "substitution of the thesis" or "substitution of the concept" will occur. While recognising the existence in thought and cognition of dialectical contradictions (such as those in the statements of opponent and reporter, defender and accuser, and so on), on the basis of its law of consistency, formal logic prohibits two opposing properties from being assigned to one and the same object, at one and the same time, and in one and the same relationship. For instance,

[1] V. I. Lenin, "Conspectus of Hegel's Book *Lectures on the Philosophy of History*", *Collected Works*, Vol. 38, p. 317.

[2] *Ibid.*, p. 182.

[3] V. I. Lenin, "Once Again on the Trade Unions, the Current Situation and the Mistakes of Trotsky and Bukharin", *Collected Works*, Vol. 32, 1979, p. 94.

formal logic does not permit the two statements "this paper is white" and "this paper is black". The law of consistency of formal logic complies fully with the law of material dialectics of the unity and struggle of opposites, for these laws operate both in everyday and scientific thinking, as well as in the education process.

Dialectical logic and formal logic are two relatively independent trends in modern logic, but they are mutually complementary and do not contradict each other. Formal logic is a necessary but not sufficient method (instrument) for cognising the truth; its laws must not be violated, while the forms of thinking it studies (concepts, judgements, propositions, conclusions) are universal. Formal logic reflects such aspects of the objective world (and thinking itself) as stability, determinacy, incompatibility of certain states and properties, and the causal links between phenomena. A knowledge of the laws and forms of correct thinking that this logic studies and compliance with them is, therefore, a necessary condition for the development of both dialectical logic and other philosophical sciences, as well as all particular sciences.

The methodological basis of the scientific interpretation of logic is provided by dialectical materialism – the science of the most general laws of development of nature, society and thinking.

Being a form of scientific logical thinking, dialectical logic, in combination with formal logic, forms the logic of scientific cognition.

The goal of cognition is to obtain true knowledge. A mastery of it is helped greatly not only by sensual cognition, but also abstract thinking. In order to make more effective use of it, people must study both materialist dialectics and its component part – subjective dialectics, as well as formal logic; they must know and apply their theories and principles in thinking, and then in their material and intellectual activities. A knowledge of the laws of dialectics and logic will help in predicting events and planning activities better, in foreseeing possible consequences, putting forward various hypotheses, teaching and studying better, seeing the "logic of things", i. e., objective dialectics, and making fuller use in practice of the knowledge obtained.

LIST OF SYMBOLS

$a \wedge b$; $a \cdot b$; $a \& b$ – conjunction "*a* and *b*".
$a \underset{\cdot}{\vee} b$, "*a* or *b*" – inclusive disjunction. (or)
$a \dot{\vee} b$; "*a* or *b*" – exclusive disjunction. = (either, or)
$a \rightarrow b$, $a \supset b$ "*a* implies *b*" ("if *a*, then *b*") – implication.
$a \equiv b$, $a \leftrightarrow b$, $a \rightleftarrows b$, $a \sim b$, "*a* is equivalent to *b*" (*a* if, and only if *b*") – equivalence.
$\bar{a}$, $\urcorner a$, $\sim a$, "not *a*" – negation of *a*.
$(\forall x)$ "for all *x*" – universal quantifier.
$(\exists x)$ "*x* exists such that" – existential quantifier.
a, b, c, ...p, q... variables for propositions.

Logic of classes

A, B, C ... variables for classes (classes *A, B, C* ...).
$\bar{A}$ – "complement of *A*".
$A \cup B$, $A + B$ – "sum of *A* and *B*".
$A \cap B$, $A \cdot B$ – "product of *A* and *B*".
$A - B$ – "difference between *A* and *B*".
$A \subset B$, $A \leqslant B$ – "*A* is included in *B*".
$a \in A$ – "element *a* belongs to class *A*".
$A \equiv B$ – "*A* is identical to *B*".

* * *

M – modal operator.
$\Box A$ – necessarily *A*.
∇A – accidentally *A*.
$\Diamond A$ – possibly *A*.
$\sim \Diamond A$ is impossible.
L_p – necessarily *p*.

$\underset{Df}{=}$ – equivalent by definition.
$\vdash$ – sign of inference.

The Polish symbols

N_x – negation of x.
Cxy – implication (x implies y).
Kxy – conjunction of x and y.
Axy – inclusive disjunction of x and y.
$[a]$ – the value of the function from argument a.
N^1x – the first negation in Post's system.
N^2x – the second negation in Post's system.
P_3 – Post's three-valued system.
$(\sim_3 p)$ – the first negation in Post's P_3 system.
$(\approx_3 p)$ – the second negation in Post's P_3 system.
$p \cdot_3 q$ – conjunction in the P_3 system.
$p \vee_3 q$ – disjunction in the P_3 system.
$p \supset_3 q$ – implication in the P_3 system.
$p \equiv_3 q$ – equivalence in the P_3 system.

Reichenbach's system

$A \supset B$ – standard implication.
$A \equiv B$ – standard equivalence.
$A \rightarrow B$ – alternative implication.
$A \ni B$ – quasi-implication.
$A \equiv B$ – alternative equivalence.
$A \cdot B$ – conjunction.
$A \vee B$ – disjunction.
$\sim A$ – cyclical negation.
$- A$ – diametrical negation.
$\bar{A}$ – complete negation.

The $G_{\chi o}$ system

$\approx_{\chi 0} p$ – negation of p.
$p \vee_{\chi 0} q$ – disjunction of p and q.
$p \wedge_{\chi 0} q$ – conjunction of p and q.
$p \supset_{\chi 0} q$ – implication of p and q.
$p \equiv_{\chi 0} q$ – equivalence of p and q.

Lewis's modal systems

$\sim p$ – negation of p.
$p \prec q$ – strict implication in Lewis's Sl system.
$\Diamond p$ – possibly p.
$p = q$ – strict equivalence.

Ackermann's system

N – the operator of necessity.
M – the operator of possibility.

$A \rightarrow B$ – Ackermann's strong implication.
λ – the logical constant ("it is absurd").
$A \& B$ – conjunction of A and B.
$\bar{A}$ – negation of A.
L – the operator of necessity in Lukasiewicz's system.
$a \mid b$ – "Sheffer's stroke" (a and b are incompatible).

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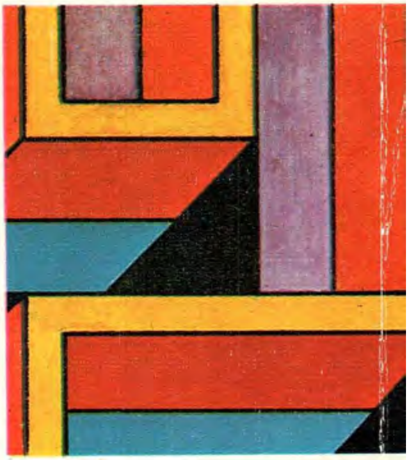

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